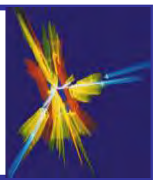
Two-particle phase space



Let's consider the simplest possible process: one particle going into two, thus study two-particle final state without specifying initial state $p=(E,\overline{p})$ except that four-momentum is conserved. Then the two-particle phase space integral is:

$$R_2(p,m_1^2,m_2^2)=\int d^4p_1 d^4p_2 \delta(p_1^2-m_1^2)\delta(p_2^2-m_2^2)\delta^4(p-p_1-p_2)$$

Note that R_2 can only be a function of s (= p^2), m_1 and m_2. First integrate over p_2 in the four-dimensional δ function imposing $p_2 = p - p_1$ and then go to the frame $p=(\sqrt{s},\overline{0})$

$$R_2(s)=\int d^4p_1\delta(p_1^2-m_1^2)\delta\{(p-p_1)^2-m_2^2\}=\int\frac{d^3p_1}{2E_1}\delta\{s-2\sqrt{s}E_1^*+m_1^2-m_2^2\}$$

$$=\int\frac{P_1^*d\Omega_1^*dE_1^*}{2}\delta\{s-2\sqrt{s}E_1^*+m_1^2-m_2^2\}=\frac{P_1^*}{4\sqrt{s}}\int d\Omega_1^*=\frac{\pi P_1^*}{\sqrt{s}}$$

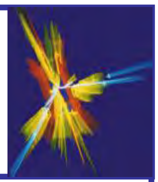
The special δ function property is used here. In addition, that $d^3p_1/2E_1=d^3p_1^*/2E_1^*$ (invariant) & $E^2=P^2+m^2 \Rightarrow EdE=PdP$ is used. Note further that the last δ integration defines

$$s-2\sqrt{s}E_1^*+m_1^2-m_2^2=0 \quad\Rightarrow\quad E_1^*=(s+m_1^2-m_2^2)/2\sqrt{s}$$

From this one obtains: $P_1^*=\sqrt{\lambda(s,m_1^2,m_2^2)}/2\sqrt{s}=P_2^*(\text{since CMF})$

So finally: $R_2(s)=\pi P_1^*/\sqrt{s}=\pi\sqrt{\lambda(s,m_1^2,m_2^2)}/2s$

One would like to be more general and derive R_2 in an arbitrary frame $p=(E,\overline{p})$ (the explicit derivation is left for the exercises). When integrating over P_2 one finds that P_1 is now given by a second order equation with two solutions denoted P_1^+ & P_1^-.



The resulting expression is:

$$R_2(E,\bar{p}) = \int \frac{d\Omega_1}{4} \sum_{\pm} \frac{(P_1^{\pm})^2}{(EP_1^{\pm} - PE_1^{\pm}\cos\theta_1)}$$

where Ω_1 is the solid angle describing the $\bar{p}_1$ orientiation & θ_1 is the angle between the momentum vectors of the decayed particle and one of the decay products. Note that this expression is valid only for timelike p ($p^2 > 0$).

Finally, let's derive the expression for lightlike p ($p^2 = 0$). The corresponding standard frame is $p = (\omega, 0, 0, \omega)$. To simplify integrals we introduce new "lightcone" variables. The parametrisation of four-vector p_1 is then given as:

$\lambda_{\pm} = E_1 \pm p_{1z} \quad \Rightarrow \quad p_1 = \{\frac{1}{2}(\lambda_+ + \lambda_-), p_{1x}, p_{1y}, \frac{1}{2}(\lambda_+ - \lambda_-)\}$. So then

$$R_2(p, m_1^2, m_2^2) = \int dp_1^0 dp_{1x} dp_{1y} dp_{1z} \delta(p_1^2 - m_1^2)\delta\{(p - p_1)^2 - m_2^2\} \quad \Rightarrow$$

$$R_2(p, m_1^2, m_2^2) = -\tfrac{1}{2}\int d\lambda_+ d\lambda_- d^2 r_1 \delta(\lambda_+\lambda_- - \bar{r}_1^2 - m_1^2)\delta\{m_1^2 - m_2^2 - 2\omega\lambda_-\},$$

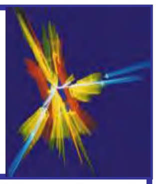
where $\bar{r}_1 = (p_{1x}, p_{1y})$ is the transverse momentum of p_1. Here we have applied a Jacobian for the transformation.

$$\frac{\partial(E_1, p_{1x}, p_{1y}, p_{1z})}{\partial(\lambda_+, \lambda_-, p_{1x}, p_{1y})} = \begin{vmatrix} \partial E_1/\partial\lambda_+ & \partial p_{1z}/\partial\lambda_+ \\ \partial E_1/\partial\lambda_- & \partial p_{1z}/\partial\lambda_- \end{vmatrix} = -\frac{1}{2}$$

Doing the integration over λ_+ and λ_- we obtain:

$$R_2(p, m_1^2, m_2^2) = \{2(m_1^2 - m_2^2)\}^{-1}\int d^2 r_1$$

Two-particle phase space



The condition for the process $p \to p_1 + p_2$ to be physical, derived from expressions for the phase space integral, is

$$\lambda(p^2, p_1^2, p_2^2) = \{p^2 - (\sqrt{p_1^2} + \sqrt{p_2^2})^2\}\{p^2 - (\sqrt{p_1^2} - \sqrt{p_2^2})^2\} \geq 0$$

If all four-vectors are timelike, the condition requires:

$$\sqrt{p^2} \geq m_1 + m_2 = \text{threshold}$$

that is a natural condition for a decay, or

$$\sqrt{p^2} \leq \left| m_1 - m_2 \right| = \text{pseudothreshold}$$

that is approriate if p is a momentum transfer.

Define symmetric Gram determinants $\Delta_n(p_1,..., p_n)$:

$$\Delta_n(p_1,..., p_n) \equiv \begin{vmatrix} p_1^2 & p_1 \cdot p_2 \cdots & p_1 \cdot p_n \\ \vdots & \vdots & \vdots \\ p_n \cdot p_1 & p_n \cdot p_2 \cdots & p_n^2 \end{vmatrix} \Rightarrow$$

$$\text{especially}: \Delta_2(p_1, p_2) = -\tfrac{1}{4}\lambda\{(p_1 + p_2)^2, p_1^2, p_2^2\}$$

Then we state that the process $p \to p_1 + p_2$ is physical if

$$\Delta_2(p_1, p_2) \leq 0$$

The boundary of the physical region in terms of invariants is obtained from the condition $\Delta_2(p_1,p_2) = 0$. Now also the λ function reveals its true significance, as an expanded form of Δ_2. One can call λ **the basic three particle kinematic function**. This follows from the fact that Δ_2 is relevant for reactions, where the total number of four-momenta is three (e.g. a $1 \to 2$ decay).

Two-particle scattering: CMF and TF relations

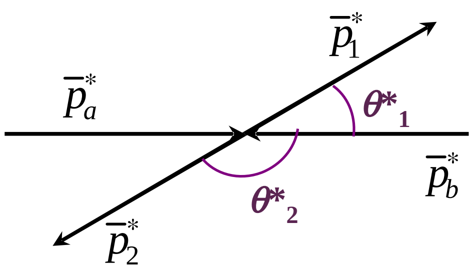
We now proceed to treat the reaction $p_a + p_b \rightarrow p_1 + p_2$. The notation used has been choosen so that it can easily be extended to higher multiplicities. The phase space (fixed by e.g. s) is 2-dimensional and can be parametrised e.g. by the scattering angle θ & one angular variable ϕ, describing rotations around beam axis. ϕ is trivial, leaving one essential final state variable. The total number of essential variables are 2: one fixes the total energy, the other the scattering angle. Use the angle between p_a and p_1 as frame-dependent angle-type variable in CMF & TF.

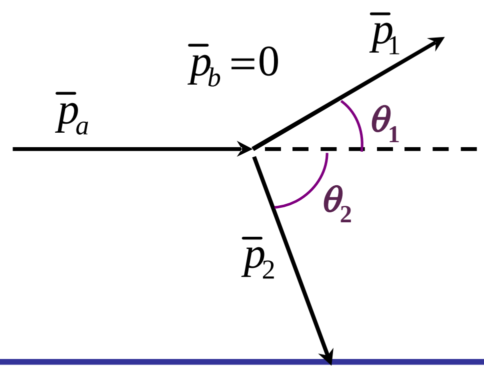
$$\theta_1^* \equiv \theta_{a1}^* = \pi - \theta_{a2}^* \qquad \theta_1 \equiv \theta_{a1}^T$$

By definition, scattering near $\theta_1^* = 0$ is called forward & scattering near $\theta_1^* = \pi$ backward. Note that the θ_1 and $\theta_2 \equiv \theta_{a2}^T$ relation is rather complicated. An invariant angle-type variable could be the invariant momentum transfer:

$$t \equiv t_{a1} = (p_a - p_1)^2 = m_a^2 - m_1^2 - 2E_aE_1 + 2P_aP_1\cos\theta_{a1}$$

In CMF, 2 $\rightarrow$ 2 scattering is kinematically very simple, since the energy– & angle–dependences are completely decoupled. This is not the case in TF. If one assumes $\sqrt{s}$ to be fixed then any one of the four final state variables P_1, θ_1, P_2 or θ_2 will determine the remaining three.





Transformation of P^* and θ^*

Let's now return to Lorentz transformations shortly to find out how P^* and θ^* transforms when going from TF to CMF. Let's investigate the sphere of constant $(P^*)^2$.

The Lorentz transformation: $p_x^* = p_x \quad p_y^* = p_y \quad p_z^* = p_z/\gamma - \beta E^* \quad \Rightarrow$

$$(P^*)^2 = \text{const.} \Rightarrow \frac{p_x^2 + p_y^2}{a^2} + \frac{(p_z - \gamma\beta E^*)^2}{b^2} = 1, \text{ where } a = P^*, b = \gamma P^*$$

This describes an ellipsoid with the distance l between focus and the centre of the ellipsoid and eccentricity ε

$$l = \sqrt{b^2 - a^2} = \gamma\beta P^* \qquad \varepsilon = l/b = \beta$$

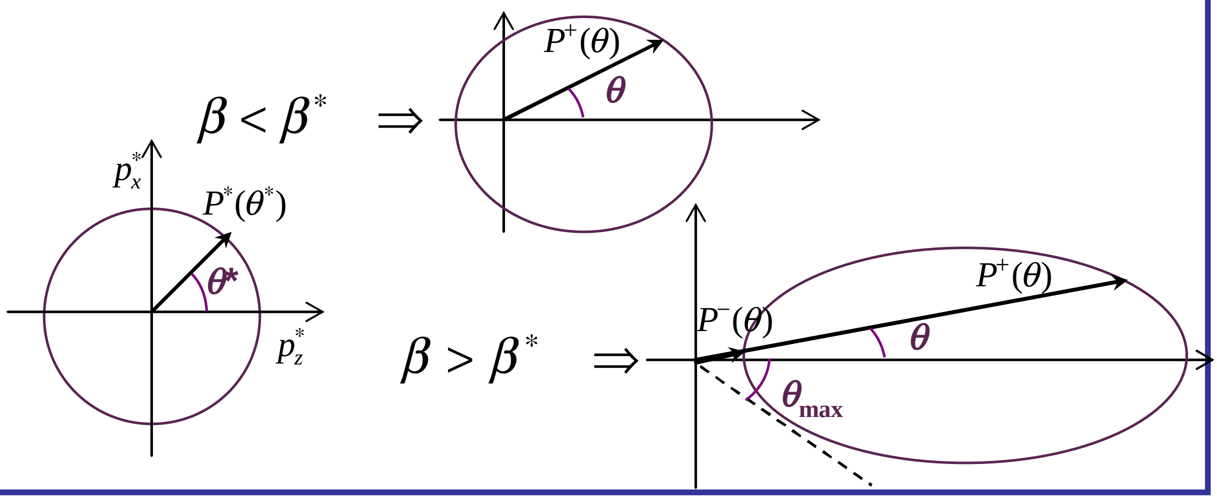
Quantitatively the results are easy to understand: the sphere's transverse dimensions remain unchanged while longitudinally it is dilated by γ & translated by $\gamma\beta E^*$. We obtain 3 different classes depending on the β–β^* ratio.

Class 1 : $\beta < \beta^*$, origin lies inside ellipsoid

Class 2 : $\beta = \beta^*$, origin lies on ellipsoid

Class 3 : $\beta > \beta^*$, origin lies outside ellipsoid

In class 3, the θ and θ^* correspondance is ambigious (2– to–1 as it is with P and P^*) & there exists a maximum angle, θ_{max}. Class 2 & 3 particles can only go in forward directions of TF while class 1 can also go backwards.



Dependence of P_i on θ_i

Since all results for P_2 can be obtained from those for P_1 by interchanging 1 and 2, it is enough to treat $P_1 = P_1(\theta_1)$. Instead of explicit derivation, let's use result of previous page. The transformation between CMF and TF was

$$\beta^{CM} = P_a \Big/ (E_a + m_b) = \sqrt{\lambda(s, m_a^2, m_b^2)} \Big/ (s - m_a^2 + m_b^2)$$

$$\gamma^{CM} = (E_a + m_b) \Big/ \sqrt{s} = (s - m_a^2 + m_b^2) \Big/ 2 m_b \sqrt{s}$$

The velocity of particle 1 and its γ parameter in CMF is:

$$\beta_1^* = P_1^* \big/ E_1^* = \sqrt{\lambda(s, m_1^2, m_2^2)} \Big/ (s + m_1^2 - m_2^2)$$

$$\gamma_1^* = E_1^* \big/ m_1 = (s + m_1^2 - m_2^2) \Big/ 2 m_1 \sqrt{s}$$

The basic parameter: $g_1^* = \dfrac{\beta^{CM}}{\beta_1^*} = \dfrac{(s + m_1^2 - m_2^2)}{(s - m_a^2 + m_b^2)} \dfrac{\sqrt{\lambda(s, m_a^2, m_b^2)}}{\sqrt{\lambda(s, m_1^2, m_2^2)}}$

Depending on whether $g_1^* < 1$ or $g_1^* \geq 1$ particle p_1 can be emitted in any direction ($0 < \theta_1 < 180^\circ$) or only in the forward hemisphere ($0 < \theta_1 < \theta_{max} \leq 90^\circ$) in TF. The $g_1^* = 1$ threshold depends on the relative magnitude of the masses. For completeness, the solutions for momentum & energy of p_1:

$$P_1^{\pm} = \left\{(E_a + m_b)^2 - P_a^2 \cos^2\theta_1\right\}^{-1} \Big[P_a \cos\theta_1 \{ m_b E_a + \tfrac{1}{2}(m_a^2 + m_b^2 + m_1^2 - m_2^2) \} \pm (E_a + m_b) [\{ m_b E_a + \tfrac{1}{2}(m_a^2 + m_b^2 - m_1^2 - m_2^2) \}^2 - m_1^2 m_2^2 - m_1^2 P_a^2 \sin^2\theta_1]^{\frac{1}{2}} \Big]$$

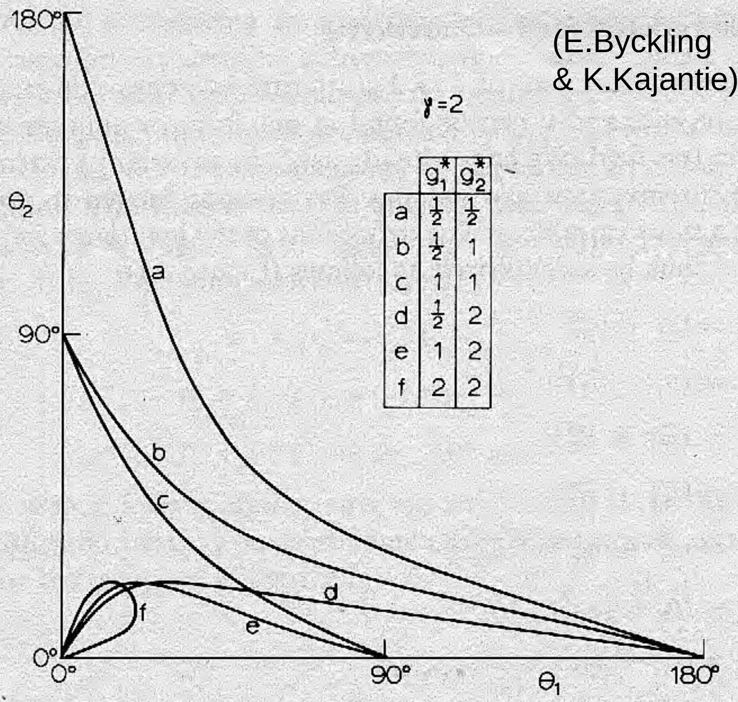
$$E_1^{\pm} = \left\{(E_a + m_b)^2 - P_a^2 \cos^2\theta_1\right\}^{-1} \Big[(E_a + m_b) \{ m_b E_a + \tfrac{1}{2}(m_a^2 + m_b^2 + m_1^2 - m_2^2) \} \pm P_a \cos\theta_1 [\{ m_b E_a + \tfrac{1}{2}(m_a^2 + m_b^2 - m_1^2 - m_2^2) \}^2 - m_1^2 m_2^2 - m_1^2 P_a^2 \sin^2\theta_1]^{\frac{1}{2}} \Big]$$

Relation between θ_1 and θ_2

Some measurements are based on coincidences. Then it is important to know θ_2, when the value of θ_1 is given. The relation can be derived in many ways. E.g. using the CMF relation $\theta_2^* = \pi - \theta_1^*$ & the Lorentz transformation of θ:

$$\tan\theta_1 = \frac{\sin\theta_1^*}{\gamma^{CM}(\cos\theta_1^* + g_1^*)} \qquad \tan\theta_2 = \frac{\sin\theta_2^*}{\gamma^{CM}(\cos\theta_2^* + g_2^*)}$$

where γ^{CM} & g_1^* are given on previous page & g_2^* can be obtained from the expression for g_1^* by interchange of 1 & 2. Now $\theta_{1,2}^*$ can be eliminated and a relation between θ_1 & θ_2 obtained. This is shown in the figure to the right:



(E.Byckling & K.Kajantie)

Figure IV.3.3 The relation between θ_1 and θ_2 for different values of g_1^* and g_2^*

As an example let's consider elastic scattering $\mu m \to \mu m$ (m target & $\mu \leq m$ beam particle mass, e.g. ep):

$$g_1^* = \frac{(s + \mu^2 - m^2)}{(s - \mu^2 + m^2)} \leq 1 \qquad g_2^* = 1$$

This implies that p_2 recoils to the forward TF hemisphere ($0 \leq \theta_2 \leq 90^\circ$) & that the beam particle can be emitted in any direction unless $\mu = m$, in which case also $\theta_1 \leq 90^\circ$.

Mandelstam variables

Let’s now introduce the whole set of invariant variables for 2 $\rightarrow$ 2 scattering, the ”Mandelstam variables”, though we have already used two of them, s and t. For reasons related to crossing one usually defines a third variable u. The definitions of the invariants for $p_a + p_b \rightarrow p_1 + p_2$ are:

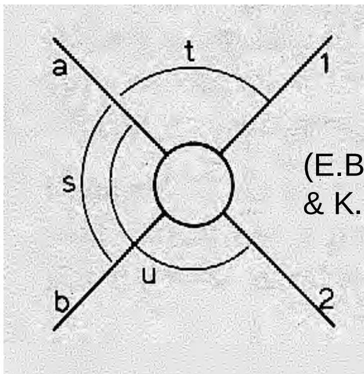
$$s = (p_a + p_b)^2 = (p_1 + p_2)^2$$
$$= (E_1^* + E_2^*)^2 = (E_a^* + E_b^*)^2 = m_a^2 + m_b^2 + 2m_b E_a^T$$

$$t = (p_a - p_1)^2 = (p_b - p_2)^2$$
$$= m_a^2 + m_1^2 - 2E_a E_1 + 2P_a P_1 \cos\theta_{a1} = m_b^2 + m_2^2 - 2m_b E_2^T$$

$$u = (p_a - p_2)^2 = (p_b - p_1)^2$$
$$= m_a^2 + m_2^2 - 2E_a E_2 + 2P_a P_2 \cos\theta_{a2} = m_b^2 + m_1^2 - 2m_b E_1^T$$

There are only two independent variables so s, t and u must be related. Infact, the relation is

$$s + t + u = m_a^2 + m_b^2 + m_1^2 + m_2^2$$



(E.Byckling & K.Kajantie)

Crossing: We have sofar threated the reaction $p_a + p_b \rightarrow p_1 + p_2$ assuming all energies are positive: $p = (E, \overline{p})$ with $E = +\sqrt{P^2 + m^2} \geq m \geq 0$. But the equation for four-momentum conservation is also analytically valid for any timelike p with a negative 0-component: $p = (E, \overline{p})$ with $E = -\sqrt{P^2 + m^2} \leq 0$. These negative energy states will in QM be seen as the positive energy states of the anti-particle.

Mandelstam variables

4-momentum conservation & antiparticle definition give:

$$
\begin{aligned}
p_a + p_b &= p_1 + p_2 & & \text{s-channel}: p_a + p_b = p_1 + p_2 \\
p_a + (-p_1) &= (-p_b) + p_2 & \Rightarrow \quad & \text{t-channel}: p_a + p_{\bar{1}} = p_{\bar{b}} + p_2 \\
p_a + (-p_2) &= p_1 + (-p_b) & & \text{u-channel}: p_a + p_{\bar{2}} = p_1 + p_{\bar{b}}
\end{aligned}
$$

where the ”bar” denotes an antiparticle & all 4-momenta now have positive E’s. For the kinematics, it is irrelevant whether a particle is an antiparticle or not but when examing dynamical properties the particle–antiparticle distinction has to be taken into account when a particle is moved from initial state to final state and vice versa.

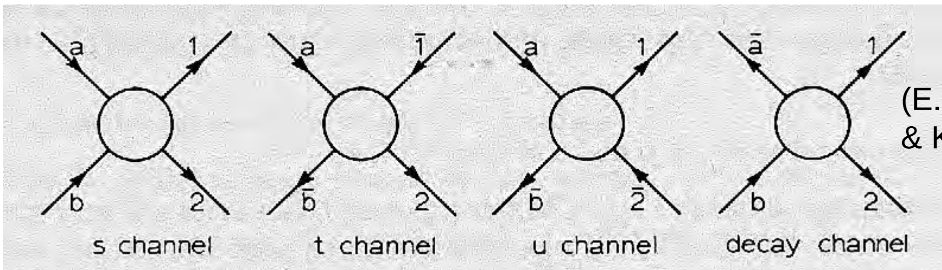


Figure IV.4.2 Various channels for $p_a + p_b \rightarrow p_1 + p_2$. The decay channel shown is physical if $m_b > m_1 + m_2 + m_a$

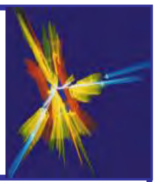
(E.Byckling & K.Kajantie)

In above equations, the channels are denoted by the invariant variable giving positive values in the channel in question. The 2 remaining possibilities are then invariant momentum transfers. E.g. t is always defined $t = (p_a - p_1)^2$ but in the t–channel p_1 has a negative E so that in frame

$$\bar{p}_a - \bar{p}_1 = \bar{p}_a + \bar{p}_{\bar{1}} = 0 \quad \Rightarrow \quad t = (E_a - E_1)^2 = (E_a + |E_1|)^2 \geq (m_a + m_1)^2$$

In addition to scattering channels, there may also exist decay channels. E.g. if $m_b \geq m_a + m_1 + m_2$ the following decay is possible $p_b \rightarrow p_{\bar{a}} + p_1 + p_2$

Relation $s,t,u \leftrightarrow$ CMF



The relations between s and CMF energies & momenta are very simple since no angles are involved. Recollect:

$$P_a^* = P_b^* = \sqrt{\lambda(s,m_a^2,m_b^2)}\Big/2\sqrt{s} \qquad P_1^* = P_2^* = \sqrt{\lambda(s,m_1^2,m_2^2)}\Big/2\sqrt{s}$$

$$E_a^* = (s+m_a^2-m_b^2)\big/2\sqrt{s} \qquad E_1^* = (s+m_1^2-m_2^2)\big/2\sqrt{s}$$

$$E_b^* = (s+m_b^2-m_a^2)\big/2\sqrt{s} \qquad E_2^* = (s+m_2^2-m_1^2)\big/2\sqrt{s}$$

The first 2 equations imply that $s \geq \max\{(m_a+m_b)^2,(m_1+m_2)^2\}$.
Introducing these expressions into the t–equation, we get

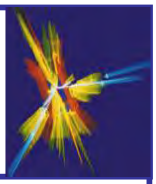
$$\begin{aligned}\cos\theta_{a1}^* &= \frac{t-m_a^2-m_1^2+2E_a^*E_1^*}{2P_a^*P_1^*} \\ &= \frac{2s(t-m_a^2-m_1^2)+(s+m_a^2-m_b^2)(s+m_1^2-m_2^2)}{\sqrt{\lambda(s,m_a^2,m_b^2)}\sqrt{\lambda(s,m_1^2,m_2^2)}} \\ &= \frac{s^2+s(2t-m_a^2-m_b^2-m_1^2-m_2^2)+(m_a^2-m_b^2)(m_1^2-m_2^2)}{\sqrt{\lambda(s,m_a^2,m_b^2)}\sqrt{\lambda(s,m_1^2,m_2^2)}} \\ &= \frac{s(t-u)+(m_a^2-m_b^2)(m_1^2-m_2^2)}{\sqrt{\lambda(s,m_a^2,m_b^2)}\sqrt{\lambda(s,m_1^2,m_2^2)}}\end{aligned}$$

Other angles can be obtained from θ_1* e.g. $\theta_2^* = \pi - \theta_1^*$.

The relations here refer to s–channel frames. Often it is more convient to be able to express quantities in t or u–channel frames in terms of s, t and u. Realized simply by permutating indices. To go from s to t–channel, swap 1 with b ($s \leftrightarrow t$, u same), to go from s to u–channel, swap 2 with b ($s \leftrightarrow u$, t same). E.g. t–channel CMF scattering angle:

$$\cos\theta_{ab}^{*(t)} = \frac{t(s-u)+(m_a^2-m_1^2)(m_b^2-m_2^2)}{\sqrt{\lambda(s,m_a^2,m_1^2)}\sqrt{\lambda(s,m_b^2,m_2^2)}}$$

Relation $s,t,u \leftrightarrow$ TF



For the initial state quantities we have already derived:

$$E_a^T = (s - m_a^2 - m_b^2)\big/2m_b \qquad P_a^T = \sqrt{\lambda(s,m_a^2,m_b^2)}\big/2m_b$$

In the final state, there are 2 energies (E_1^{T}, E_2^{T}), momenta (P_1^{T}, P_2^{T}) and angles (θ_1^{T}, θ_2^{T}) to determine. The energies are now most simply related to momentum transfers:

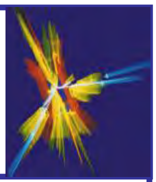
$$E_2^T = (m_b^2 + m_2^2 - t)\big/2m_b \quad \Rightarrow \quad P_2^T = \sqrt{\lambda(t,m_b^2,m_2^2)}\big/2m_b$$

$$E_1^T = (m_b^2 + m_1^2 - u)\big/2m_b \quad \Rightarrow \quad P_1^T = \sqrt{\lambda(u,m_b^2,m_1^2)}\big/2m_b$$

The equations above require the validity of the pseudo-threshold: $t \le (m_b - m_2)^2, u \le (m_b - m_1)^2$ for the momenta P_1^{T} and P_2^{T} to be real. Finally, the angles θ_1^{T} and θ_2^{T} can be obtained from expanding $t = (p_a - p_1)^2$ and $u = (p_a - p_2)^2$:

$$\cos\theta_{a1}^T = \frac{(s - m_a^2 - m_b^2)(m_b^2 + m_1^2 - u) + 2m_b^2(t - m_a^2 - m_1^2)}{\sqrt{\lambda(s,m_a^2,m_b^2)}\sqrt{\lambda(u,m_b^2,m_1^2)}}$$

$$\cos\theta_{a2}^T = \frac{(s - m_a^2 - m_b^2)(m_b^2 + m_2^2 - t) + 2m_b^2(u - m_a^2 - m_2^2)}{\sqrt{\lambda(s,m_a^2,m_b^2)}\sqrt{\lambda(t,m_b^2,m_2^2)}}$$



The reaction cross section for $p_a + p_b \to p_1 + p_2$ is:

$$\sigma(s) = \frac{1}{8\pi^2 \sqrt{\lambda(s, m_a^2, m_b^2)}} \int \frac{d^3\overline{p}_1}{2E_1} \frac{d^3\overline{p}_2}{2E_2} \delta^4(p_a + p_b - p_1 - p_2) |\mathrm{M}|^2$$

The matrix element M depends here on two independent variables (e.g. an invariant and an angle). If a differential cross section $d\sigma/dx$ is computed, no further integration over M is necessary (since one nontrivial variable is still left).

Doing the integration over phase space partly, one gets:

$$\frac{d\sigma}{d\Omega_1^*} = \frac{|\mathrm{M}|^2 \sqrt{\lambda(s, m_1^2, m_2^2)}}{64\pi^2 s \sqrt{\lambda(s, m_a^2, m_b^2)}} \quad \text{(in CMF, derivation in exercises)}$$

$$\frac{d\sigma}{d\Omega_1^T} = \frac{|M|^2}{64\pi^2 m_b P_a^T} \cdot \frac{(P_1^T)^2}{(E_a^T + m_b) P_1^T - P_a^T E_1^T \cos\theta_{a1}^T} \quad \text{(in TF)}$$

Similar formulas for $d\sigma/d\Omega_2{}^*$ & $d\sigma/d\Omega_2{}^{\mathrm{T}}$ obtained by inter-changing 1 ↔ 2. In many cases, it is more convient to have an invariant cross section like $d\sigma/dt$ than the above:

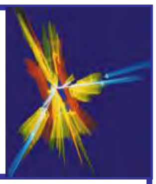
$$\frac{d\sigma}{dt} = \frac{|\mathrm{M}|^2}{16\pi\lambda(s, m_a^2, m_b^2)} \quad \text{(in any frame, derivation in exercises)}$$

The reaction cross section can now be determined

$$\sigma(s) = \frac{1}{16\pi\lambda(s, m_a^2, m_b^2)} \int_{t^-}^{t^+} dt |\mathrm{M}(s, t)|$$

where $t^{\pm}$ (s, m_i^2) are the limits on t for fixed s (see later).

Optical theorem



The **optical theorem** relates the **total cross section** for the process $p_a + p_b \to$ anything with the forward scattering amplitude of the corresponding elastic process (see e.g. G. Källen: Elementary particle physics, Addison–Wesley, 1964):

$$\mathrm{Im}\{\, \mathrm{M}_{\text{elastic}}\,(s,t=0)\} = \sqrt{\lambda(s,m_a^2,m_b^2)}\,\sigma_{tot}(s)$$

$$\text{plus} \qquad \frac{d\sigma}{dt} = \frac{|\mathrm{M}|^2}{16\pi\lambda(s,m_a^2,m_b^2)} \quad \Rightarrow$$

$$(\mathrm{Re}\{\, \mathrm{M}_{\text{elastic}}\,(s,t=0)\})^2 = \lambda(s,m_a^2,m_b^2)\left(16\pi \left.\tfrac{d\sigma}{dt}\right|_{t=0} - \sigma_{tot}^2(s)\right)$$

Implies that the "optical point" is related to the total cross section: $\left.\dfrac{d\sigma}{dt}\right|_{t=0} \geq \dfrac{1}{16\pi}\sigma_{tot}^2(s)$

The TOTEM experiment at LHC utilizes the optical theorem to determined the total *pp* cross section in a luminosity independent way by combining the differential distribution *dN*/*dt* determination of elastic scattering at t = 0 (very forward scattering) with a counting experiment. The *dN*/*dt* at t = 0 is determined from extrapolating the measured distribution at small t (~2–3 · 10^{-3} GeV^2) to t = 0.

$$\left.\begin{aligned} L\sigma_{tot}^2 &= \frac{16\pi}{1+\rho^2}\times\left.\frac{dN}{dt}\right|_{t=0} \\ L\sigma_{tot} &= N_{elastic} + N_{inelastic} \end{aligned}\right\} \Longrightarrow \boxed{\sigma_{tot} = \frac{16\pi}{1+\rho^2}\times\frac{(dN/dt)|_{t=0}}{N_{el}+N_{inel}}}$$

$$\rho \equiv \mathrm{Re}\{\mathrm{M}_{\text{elastic}}(s,t=0)\}/\mathrm{Im}\{\mathrm{M}_{\text{elastic}}(s,t=0)\}$$

$$\approx 0.13\text{-}0.14 \text{ at LHC energies} (\sqrt{s} = 14\,\mathrm{TeV})$$

Physical region in s,t,u

When the reaction $p_a + p_b \to p_1 + p_2$ is described by E_1* & θ_{a1}*, then the physical region for the s–channel can be defined by $E_1^* \geq m_1$, $-1 \leq \cos\theta_{a1}^* \leq 1$. In other words, the reaction $p_a + p_b \to p_1 + p_2$ can experimentally have any values satisfying these conditions. Let's express the physical reaction region in the plane of two invariants (e.g. s & t). For simplicity, we start from elastic scattering.

Let's analyse the reaction $p_a + p_b \to p_1 + p_2$ when $m_a = m_1 = \mu$ and $m_b = m_2 = m$, assume $\mu \leq m$. CMF quantities are:

$$E_a^* = E_1^* = (s + \mu^2 - m^2)\big/2\sqrt{s} \qquad E_b^* = E_2^* = (s + m^2 - \mu^2)\big/2\sqrt{s}$$

$$P_a^* = P_b^* = P_1^* = P_2^* \equiv P^* = \sqrt{\lambda(s,\mu^2,m^2)}\big/2\sqrt{s}$$

Definition of t gives a relation for the scattering angle θ_{a1}*:

$$\cos\theta_{a1}^* = \frac{t - m_a^2 - m_1^2 + 2E_a^*E_1^*}{2P_a^*P_1^*} = 1 + \frac{2st}{\lambda(s,\mu^2,m^2)}, \quad \text{or}$$

$$t = -\frac{\lambda(s,\mu^2,m^2)}{2s}\left(1 - \cos\theta_{a1}^*\right) = -4(P^*)^2\sin^2(\theta_{a1}^*/2)$$

Definition of u gives a relation for the scattering angle θ_{a2}*:

$$\cos\theta_{a2}^* = \frac{u - m_a^2 - m_2^2 + 2E_a^*E_2^*}{2P_a^*P_2^*} = 1 + \frac{2su - 2(m^2 - \mu^2)^2}{\lambda(s,\mu^2,m^2)}$$

now $\theta_{a2}^* = \pi - \theta_{a1}^*$, so $u = \frac{(m^2 - \mu^2)^2}{s} - \frac{\lambda(s,\mu^2,m^2)}{2s}\left(1 + \cos\theta_{a1}^*\right)$

This implies the following conditions for backward/forward direction:

$$u(\theta_{a1}^* = \pi) = \frac{(m^2 - \mu^2)^2}{s} \qquad t(\theta_{a1}^* = 0) = 0$$

Physical region in *s,t,u*

The requirement $-1 \le \cos\theta_{a1}^* \le 1$ now gives the boundary of the physical region. The latter condition gives a straight line in the *st*–plane $t = 0, \quad u = 2m^2 + 2\mu^2 - s \quad (\cos\theta^*_{a1} = 1)$

& the former a hyperbola: $t = -\dfrac{\lambda(s,m^2,\mu^2)}{s}, \quad u = \dfrac{(m^2-\mu^2)^2}{s} \quad (\cos\theta^*_{a1} = -1)$

with the asymptotes $s = 0, \quad u = 0 \;(\text{or } t = 2m^2 + 2\mu^2 - s)$. The curves intersect at $s = (m\pm\mu)^2$. $s = (m+\mu)^2$ corresponds to the threshold ($P^* = 0$) of the reaction. Although only the *s*–channel was used in derivation, we obtain the physical regions also for the *u*– ($u \ge (m+\mu)^2$) & *t*–channel ($t \ge 4m^2$).

(E.Byckling & K.Kajantie)

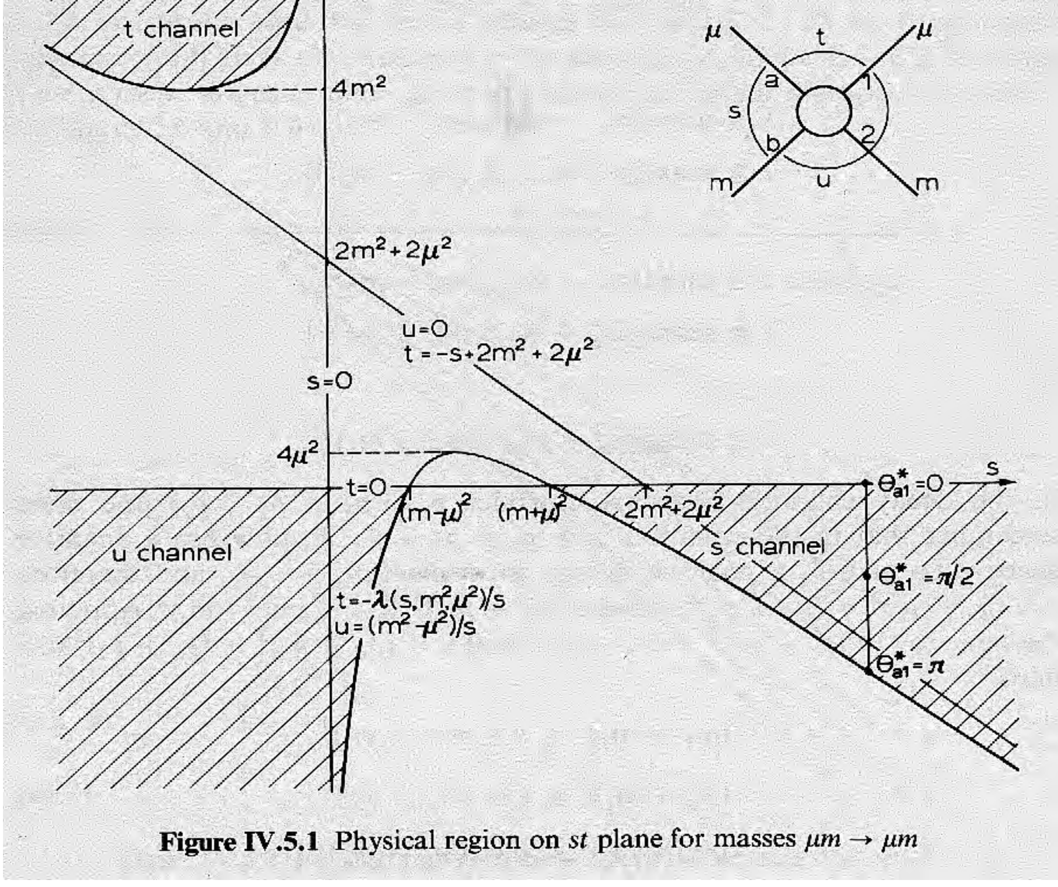


Figure IV.5.1 Physical region on *st* plane for masses $\mu m \rightarrow \mu m$

Physical region in s,t,u

We can now return to the case of arbitrary masses. By considering a decay $p \to p_i + p_j$, we proved earlier that

$$\Delta_2(p_i, p_j) = p_i^2 p_j^2 - (p_i \cdot p_j)^2 \leq 0 \quad \Leftrightarrow$$
$$(p_i + p_j)^2 \geq (m_i + m_j)^2 \quad \text{or} \quad (p_i + p_j)^2 \leq (m_i - m_j)^2$$

Logically then in $p_a + p_b \to p_1 + p_2$, $s = (p_1+p_2)^2 = (p_a+p_b)^2$ has to be larger than both $(m_1+m_2)^2$ & $(m_a+m_b)^2$ or smaller than both $(m_1-m_2)^2$ & $(m_a-m_b)^2$. Same will be valid for other invariants so physical s, t & u in scattering have to satisfy:

$$s \geq \max\{(m_a+m_b)^2, (m_1+m_2)^2\} \quad \text{or}$$
$$s \leq \min\{(m_a-m_b)^2, (m_1-m_2)^2\}, t \geq \max\{(m_a+m_1)^2, (m_b+m_2)^2\} \quad \text{or}$$
$$t \leq \min\{(m_a-m_1)^2, (m_b-m_2)^2\}, u \geq \max\{(m_a+m_2)^2, (m_b+m_1)^2\} \quad \text{or}$$
$$u \leq \min\{(m_a-m_2)^2, (m_b-m_1)^2\}$$

In decay channels (e.g. $p_b \to p_{\bar{a}} + p_1 + p_2$), the fact that $p_{\bar{a}}$ has negative energy has to be taken into account. So $s = (p_1+p_2)^2 = (p_b-p_a)^2$ has to satisfy $(m_1+m_2)^2 \leq s \leq (m_b-m_a)^2$.

$$(m_1+m_2)^2 \leq s = (p_b - p_a)^2 = (p_1 + p_2)^2 \leq (m_b - m_a)^2$$
$$(m_a+m_1)^2 \leq t = (p_a + p_1)^2 = (p_b - p_2)^2 \leq (m_b - m_2)^2$$
$$(m_a+m_2)^2 \leq u = (p_b - p_1)^2 = (p_a + p_2)^2 \leq (m_b - m_1)^2$$

These inequalities are satisfied only if $m_b \geq m_a + m_1 + m_2$. Similar conditions are obtained in other decay channels.

In addition to the above, $|\cos\theta_{a1}^*| \leq 1$ has to be imposed, which will be shown to lead to a condition of the type $\Delta_3 \geq 0$ and which obviously restricts s and t simultaneously.

Physical region in *s,t,u*

The easiest way to see the effect of the $\cos\theta_{a1}^*$ constraint is to insert $\cos\theta_{a1}^* = \pm 1$ in e.g. the definition of t:

$$t^{\pm} = m_a^2 + m_1^2 - \frac{1}{2s}\{(s + m_a^2 - m_b^2)(s + m_1^2 - m_2^2) - \pm\sqrt{\lambda(s,m_a^2,m_b^2)}\sqrt{\lambda(s,m_1^2,m_2^2)}\}$$

where the $\pm$ refers to the two $\cos\theta_{a1}^*$ values. Especially, t^+ (often called $|t|^{\min}$) is the value of t in the forward direction. In general, the above equation gives the boundary of the physical region for $2 \to 2$ scattering in the st plane. To see the symmetries between the various channels and obtain the result in compact form, it is most convient to fix the boundary from the condition $\sin\theta_{a1}^* = 0 \Leftrightarrow \cos\theta_{a1}^* = \pm 1$.

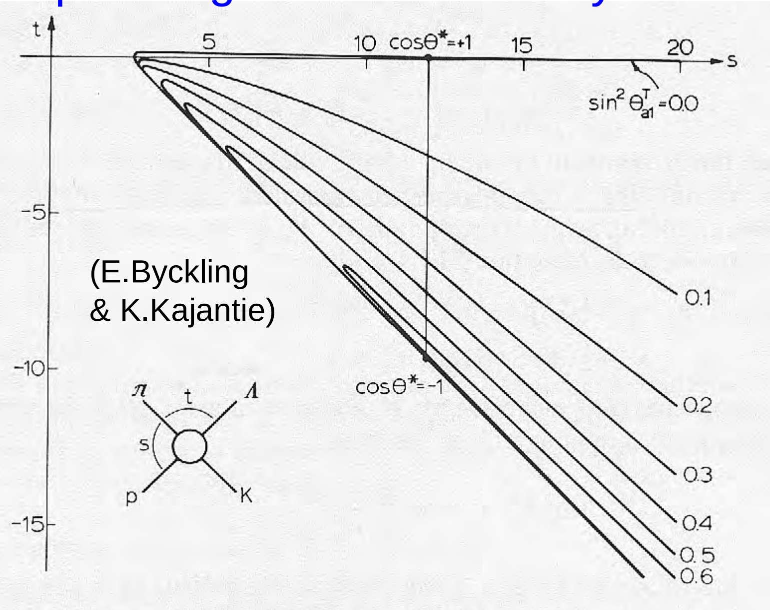


Figure IV.5.2 Physical region of $\pi p \to \Lambda K$ in the st plane. Curves of constant target system scattering angle θ_{a1}^{T} are also shown

$$\Delta_3(p_a,p_b,p_1) \equiv \begin{vmatrix} p_a^2 & p_a\cdot p_b & p_a\cdot p_1 \\ p_b\cdot p_a & p_b^2 & p_b\cdot p_1 \\ p_1\cdot p_a & p_1\cdot p_b & p_1^2 \end{vmatrix} = s(P_a^*)^2(P_1^*)^2\sin^2\theta_{a1}^*$$

From the determinant one obtains **the basic four-particle kinematic function** $G(x, y, z, u, v, w)$, which corresponds to Δ_3 in the same way as $\lambda(x, y, z)$ corresponds to Δ_2 :

$$\Delta_3(p_a,p_b,p_1) = -\tfrac{1}{4}G\{(p_a+p_b)^2,(p_a-p_1)^2,(p_a+p_b-p_1)^2,p_a^2,p_b^2,p_1^2\}$$
$$= -\tfrac{1}{4}G(s,t,m_2^2,m_a^2,m_b^2,m_1^2)$$

Physical region in *s,t,u*

So then $\sin^2\theta^*_{a1} = -4s\dfrac{G(s,t,m_2^2,m_a^2,m_b^2,m_1^2)}{\lambda(s,m_a^2,m_b^2)\lambda(s,m_1^2,m_2^2)}$

The physical region for 2 → 2 scattering in the *st* plane has to satisfy in addition to the previous mass conditions, the requirement: $\Delta_3 \geq 0$ where the arguments may be any three linearly independent combinations of p_a, p_b, p_1 and p_2. An equivalent requirement based on the G–function is:

$$G(s,t,m_2^2,m_a^2,m_b^2,m_1^2) \leq 0$$

(E.Byckling & K.Kajantie)

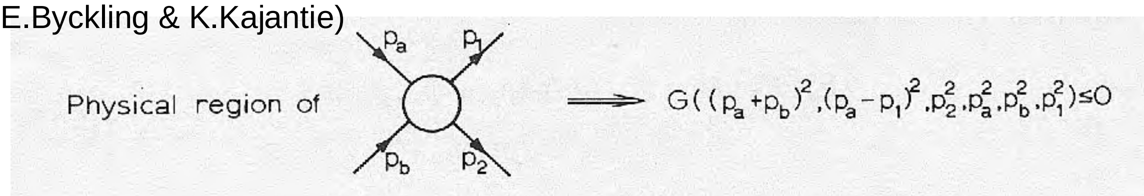


Figure IV.5.3

NB! valid even if some of the p_i's are groups of particles.

The algebraic expression for $G(x, y, z, u, v, w)$ is:

$$G(x,y,z,u,v,w) = x^2y + xy^2 + z^2u + zu^2 + v^2w + vw^2 + xzw$$
$$+xuv + yzv + yuw - xy(z+u+v+w)$$
$$-zu(x+y+v+w) - vw(x+y+z+u)$$

If $m_a = m_1 = \mu$ and $m_b = m_2 = m$ (elastic scattering):

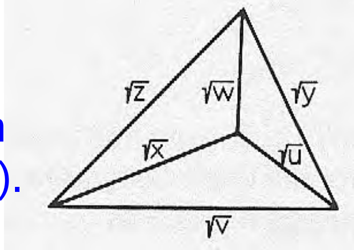
$$G(x,y,z,u,z,u) = y\{xy + \lambda(x,z,u)\} \quad \Rightarrow \quad \sin^2\theta^*_{a1} = -\frac{4st\{st+\lambda(s,m^2,\mu^2)\}}{\{\lambda(s,m^2,\mu^2)\}^2}$$

(equivalent with our previous result for $\cos\theta_{a1}^*$)

NB! The physical region of symmetric Gram determinants depends on the order, e.g. $\Delta_2 \leq 0$ and $\Delta_3 \geq 0$. The rule is that odd orders ≥ 0 and even ≤ 0. Implies e.g. that for 2 → 3 scattering, the physical region is described by $\Delta_4 \leq 0$.

Physical region in *s,t,u*

(E.Byckling & K.Kajantie)



Exactly as $\lambda(x, y, z)$ is related to the area of a triangle, one can see that $G(x, y, z, u, v, w)$ = (–144)∗(squared volume of a tetrahedron with pairwise opposite sides $\sqrt{x},\sqrt{y};\sqrt{z},\sqrt{u};\sqrt{v},\sqrt{w}$). G is also called the tetrahedron function.

G is invariant under certain permutations of its six arguments namely under any permutations of the four faces of the tetrahedron. If we group the arguments of G into three groups xy, zu & vw corresponding to opposite sides of the tetrahedron. Then G is invariant under (i) any permutation of these groups, (ii) any simultaneous interchange of arguments inside the groups.

G often needed as a function of two variables, e.g. s & t. Due to symmetry properties there are only two essentially different cases, xy and xz (or permutations). G = 0 is 3rd order in x, y & 2nd order in x, z. The 2nd order solutions are

$$x^{\pm} = z + w - \frac{1}{2y}\left\{(y+z-v)(y+w-u) - \pm\sqrt{\lambda(y,z,v)}\sqrt{\lambda(y,u,w)}\right\}$$
$$= u + v - \frac{1}{2y}\left\{(y-z+v)(y-w+u) - \pm\sqrt{\lambda(y,z,v)}\sqrt{\lambda(y,u,w)}\right\}$$

The solutions for y and z are obtained from $x^{\pm}$ by the permutations $x \leftrightarrow y$, $z \leftrightarrow u$ and $x \leftrightarrow z$, $u \leftrightarrow y$, respectively. We always want $x^{+} \geq x^{-}$ & the sign determined when $y > 0$. The equations above can be understood as analogues of previous equations evaluated for $\cos\theta_{a1}{}^{*} = \pm 1$. The ± signs corresponds to cosine = ±1, evaluated in the rest frame of the variable dividing the brackets, e.g. CMF if $y = s$. The λ's are related to momenta & the first terms to energies.

Physical region in *s,t,u*

E.g. for $p_a + p_b \to p_1 + p_2$, we get the following solutions:

$$y^{\pm} = u + w - \frac{1}{2x}\left\{(x+u-v)(x+w-z) - \pm\sqrt{\lambda(x,u,v)}\sqrt{\lambda(x,z,w)}\right\} \Rightarrow$$

$$t^{\pm} = m_a^2 + m_1^2 - \frac{1}{2s}\left\{(s+m_a^2-m_b^2)(s+m_1^2-m_2^2) - \pm\sqrt{\lambda(s,m_a^2,m_b^2)}\sqrt{\lambda(s,m_2^2,m_1^2)}\right\}$$

(identical to our previous result)

Note that G now can be expressed in terms of its roots:

$$G(x,y,z,u,v,w) = y(x-x^+)(x-x^-) = x(y-y^+)(y-y^-) \quad \text{etc...}$$

Let's rewrite G in such a way that the symmetry between s, t and u–channels (& hence boundaries) is explicitly seen

$$-G(x,y,z,u,v,w) \equiv \Phi(s,t) = stu - (\alpha s + \beta t + \gamma u), \text{ where}$$

$$K\alpha = (m_a^2 m_b^2 - m_1^2 m_2^2)(m_a^2 + m_b^2 - m_1^2 - m_2^2)$$
$$K\beta = (m_a^2 m_1^2 - m_b^2 m_2^2)(m_a^2 + m_1^2 - m_b^2 - m_2^2)$$
$$K\gamma = (m_a^2 m_2^2 - m_b^2 m_1^2)(m_a^2 + m_2^2 - m_b^2 - m_1^2)$$
$$K = m_a^2 + m_b^2 + m_1^2 + m_2^2 = s + t + u$$

It is also convient to use tri-angular coordinates. The axes are then three lines intersecting at 60°, and s, t, u are the distances from corresponding axis. When the height of the triangle between the lines is Σm_i^2, the condition $s + t + u = \Sigma m_i^2$ is automatically satisfied.

(E.Byckling & K.Kajantie)

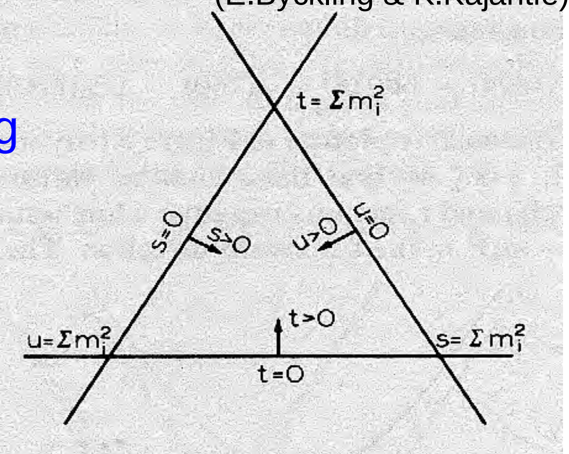
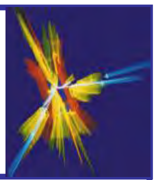


Figure IV.5.5 Triangular coordinates for *s, t, u*

Physical region in s,t,u



The cubic curve $\Phi(s,t) = 0$ has the following properties:

1. The asymptotes are $s = 0$, $t = 0$, $u = 0$.

2. The curve $\Phi(s,t) = 0$ intersects the asymptotes in the following three points on the line $\alpha s + \beta t + \gamma u = 0$:

$$s = 0: \quad t = -\gamma K/(\beta-\gamma) \quad u = \beta K/(\beta-\gamma)$$
$$t = 0: \quad s = -\gamma K/(\alpha-\gamma) \quad u = \alpha K/(\alpha-\gamma)$$
$$u = 0: \quad s = -\beta K/(\alpha-\beta) \quad t = \alpha K/(\alpha-\beta)$$

3. The tangents of $\Phi(s,t) = 0$ parallell to the three coordinate axes are the following twelve lines at threshold and pseudothreshold in the 3 channels:

$$s = (m_a \pm m_b)^2, \quad s = (m_1 \pm m_2)^2, \quad t = (m_a \pm m_1)^2$$
$$t = (m_b \pm m_2)^2, \quad u = (m_a \pm m_2)^2, \quad u = (m_b \pm m_1)^2$$

To make this a bit more practical let's consider a reaction $\pi^0(135) + \omega(783) \to \pi^+(140) + \pi^-(140)$. Now $m_a + m_b > m_1 + m_2$ so that the s–channel threshold is $(m_a + m_b)^2 \approx 0.843$ GeV^2. In the reaction $K \approx 0.671$ GeV^2, $K\alpha \approx 6.4 \cdot 10^{-3}$ GeV^6 and $K\beta = K\gamma \approx 6.9 \cdot 10^{-3}$ GeV^6. Since γ & $\beta > \alpha$, the curve $\Phi(s,t) = 0$ intersects with the asymptotes $t = 0$ & $u = 0$ at an s–value that is positive (NB! no intersection with $s = 0$ since $\gamma = \beta$).

In the s–channel, t and u may attain $(m_a - m_1)^2$ and $(m_a - m_2)^2$ respectively (both $\approx 2 \cdot 10^{-5}$ GeV^2). The t–channel is $\pi^0 + \pi^- \to \omega + \pi^-$ so the threshold is $(m_b + m_2)^2 \approx 0.852$ GeV^2. Both s & u remain negative. Same threshold value and s & t behaviour holds also for the u–channel $\pi^0 + \pi^+ \to \omega + \pi^+$.

The remaining pseudothresholds and thresholds are tangents to a connected central region, which lies in the region where s, t & u are all positive. In case there are thresholds that are smaller than the parallell pseudo-thresholds (↔ one mass is larger than the sum of the three other), this region represents a particle decay. If this is not so, the region is unphysical ($\Delta_3 \geq 0$ but not $\Delta_2 \leq 0$).

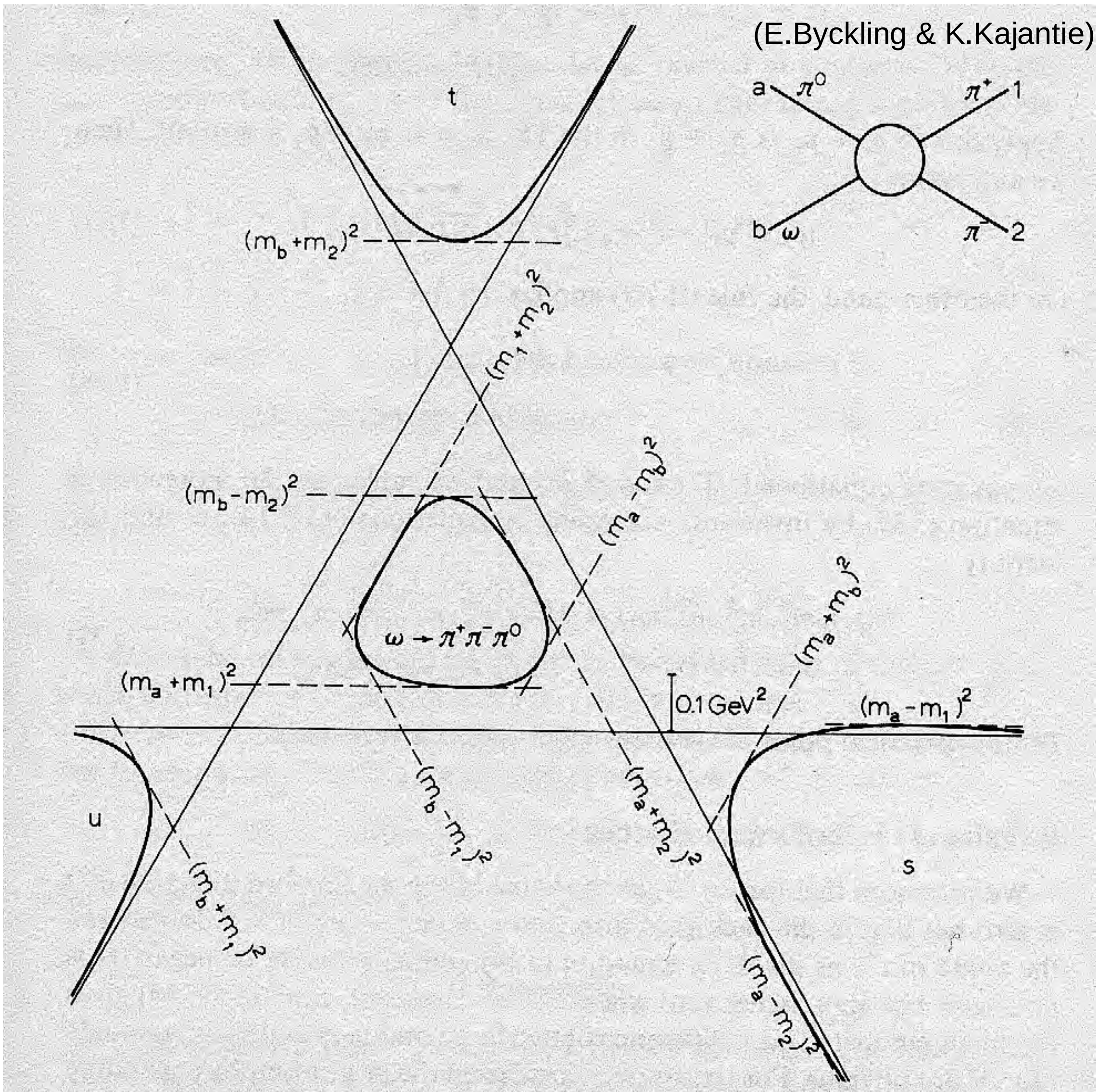


Figure IV.5.7 Physical regions of the reaction $\pi^0\omega \to \pi^+\pi^-$. The central region is now physical (Dalitz plot). The s channel boundary is distorted as in Figure 5.6

t-value in forward direction

We have seen that for $\mu m \to \mu m$, the value of t in the forward direction (t^+) is 0 but in the backward direction $u = (m^2 - \mu^2)^2/s > 0$. In general, $t^+ \neq 0$ but approaches 0 when $s \to \infty$. To obtain approximate value, let´s expand the λ–functions for large s in the expression for t^+.

$$\text{e.g.}\quad \sqrt{\lambda(s,m_a^2,m_b^2)} = \sqrt{(s-m_a^2-m_b^2)^2 - 4m_a^2m_b^2} = (s-m_a^2-m_b^2)\sqrt{1-4\varepsilon_{ab}^2}$$

$$\approx (s-m_a^2-m_b^2)(1-2\varepsilon_{ab}^2), \ \text{where} \quad \varepsilon_{ab} = \frac{m_a m_b}{s-m_a^2-m_b^2} \propto \frac{1}{s} \quad \Rightarrow$$

$$t^+ = m_a^2 + m_1^2 - \frac{(s+m_a^2-m_b^2)(s+m_1^2-m_2^2) - \sqrt{\lambda(s,m_a^2,m_b^2)}\sqrt{\lambda(s,m_1^2,m_2^2)}}{2s}$$

$$= -\frac{(m_a^2-m_1^2)(m_b^2-m_2^2)}{s} + O\left(s^{-2}\right) \quad \overset{m_a \neq m_1 \text{ and } m_b \neq m_2}{\Rightarrow}$$

$$t^+ \approx -\frac{(m_a^2-m_1^2)(m_b^2-m_2^2)}{s} \qquad (m_a m_b \to m_1 m_2)$$

So t^+ is positive if $m_a < m_1 \wedge m_b > m_2$ or $m_a > m_1 \wedge m_b < m_2$. For instance $\mu m \to m\mu$ ($m_a = m_2 = \mu$, $m_b = m_1 = m$) gives:

$$t^+ \approx \frac{(m^2-\mu^2)^2}{s} \qquad (\mu m \to m\mu)$$

that happens to be the same result as for the backward value of u in $\mu m \to \mu m$ ($\leftrightarrow$ forward t^+ value in $\mu m \to m\mu$).

Note that if either $m_a = m_1$ or $m_b = m_2$, t^+ is $\propto 1/s^2$. An example is resonance production $\mu m \to \mu m^*$ ($m^* > m$) :

$$t^+ \approx -\frac{\mu^2(m^{*2}-m^2)^2}{s^2} < 0 \qquad (\mu m \to \mu m^*)$$