Monte Carlo Method

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I SPRACE Physics Analysis Workshop
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Motivation

- Evaluation of cross sections leads to

$$\sigma = \int dx_1 dx_2 \sum_{\text{subp}} f_{a_1/p}(x_1)\, f_{a_2/\bar{p}}(x_2)$$

$$\frac{1}{2\hat{s}(2\pi)^{3n-4}} \int d\Phi_n(x_1 P_A + x_2 P_B;\ p_1 \ldots p_n)\Theta(\text{cuts})\overline{\sum}|\mathcal{M}|^2(a_1 a_2 \to b_1 \ldots b_n)$$

there are 3n-2 integrals. We also need to simulate the detector!

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- We need effective techniques to perform the calculations!

Shortcomings of traditional numerical methods

- Traditional methods work well for low dimensional integrals:

method/uncertainty	1 dimension	d dimensions
Trapedoidal rule	$\frac{1}{n^2}$	$\frac{1}{n^{2/d}}$
Simpson's rule	$\frac{1}{n^4}$	$\frac{1}{n^{4/d}}$
Gauss rule	$\frac{1}{n^{2m-1}}$	$\frac{1}{n^{(2m-1)/d}}$
Monte Carlo	$\frac{1}{\sqrt{n}}$	$\frac{1}{\sqrt{n}}$

Introduction

- MC is a stochastic technique
- MC provide approximate solutions using statistical sampling experiments.
- MC has a wide range of applications from economics to physics
- MC is a statistical method used in simulation of data
- MC uses a sequence of random numbers as data
- MC can be applied to problems with no probabilistic content

Basic idea

- MC is the most efficient way to perform multi-dimensional integrals.
- The simplest idea: integrand is a function of a random variable

$$x \in [0,1] \qquad \text{and} \qquad \langle f \rangle = \int_0^1 dx\, f(x) \simeq \frac{1}{N} \sum_{j=1}^{N} f(x_j)$$

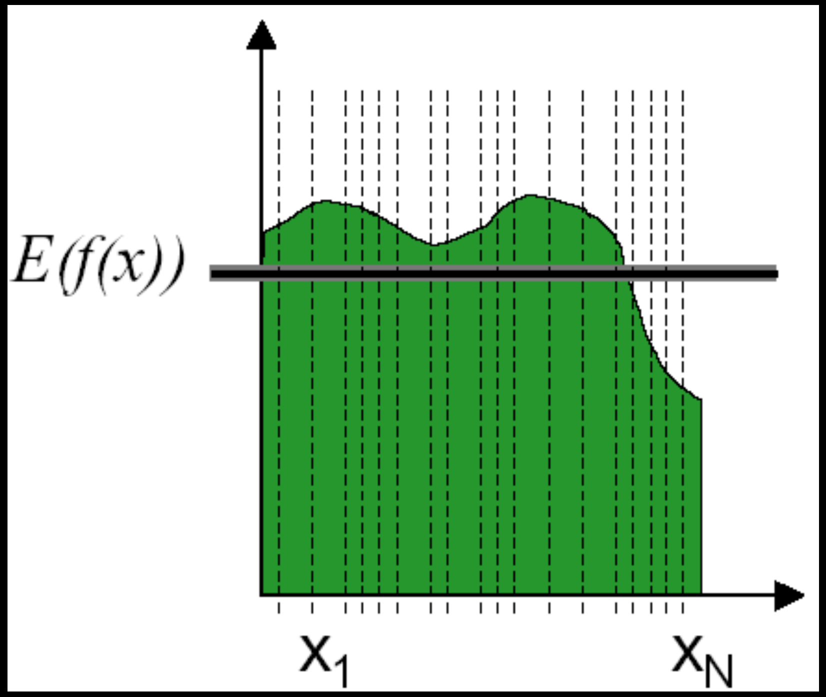
x is uniformly distributed [crude MC]

- f(x) is a crude estimator of $\langle f \rangle$
- f(x) is a random variable with variance

$$\sigma_1^2 = \int_0^1 dx\, (f - \langle f \rangle)^2 \qquad \Longrightarrow \qquad \sigma_N = \frac{\sigma_1}{\sqrt{N}}$$

- we can estimate the error from the MC simulation

$$s^2 = \frac{1}{n-1}\sum_{j=1}^{n}(f(x_j) - \langle f \rangle)^2$$



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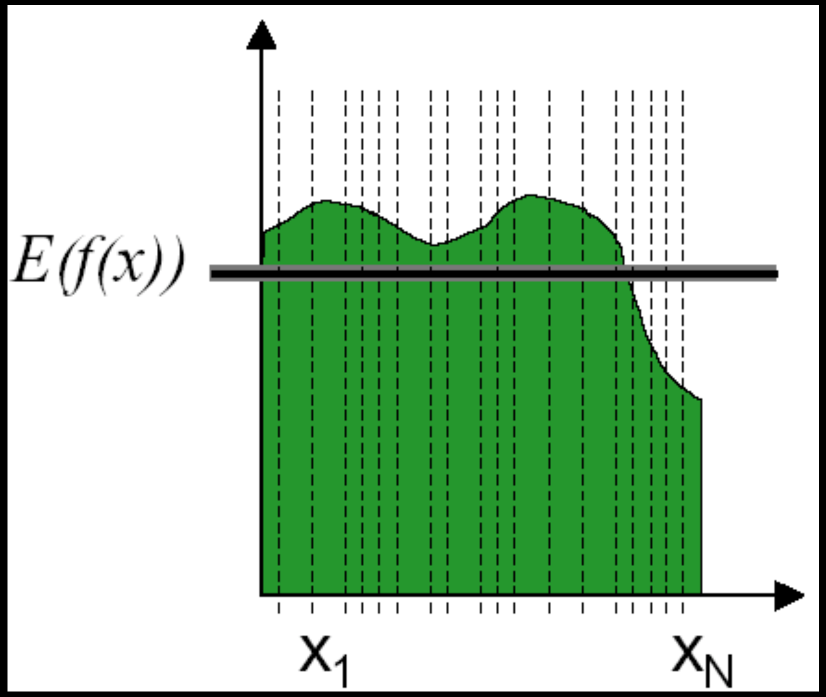


Initial remarks

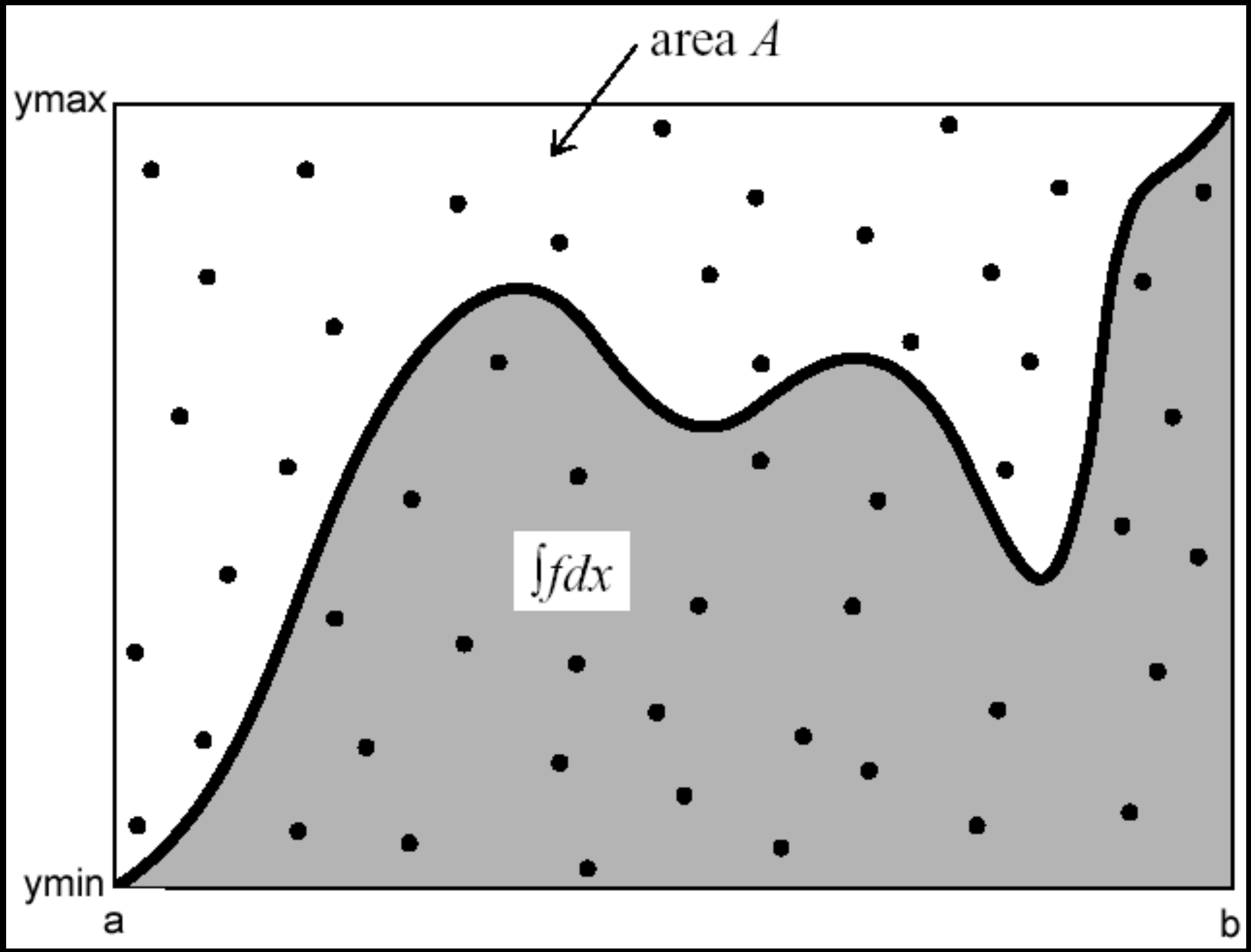
1. MC is exact for f constant. The flatter the better!
2. We should avoid near-singular integrands, e.g.,

$$\int \frac{ds}{(s-M)^2 + M^2\Gamma^2} = \frac{d\theta}{M\Gamma} \qquad \text{with} \qquad s - M^2 = M\Gamma\tan\theta$$

3. Avoid discontinuities of f if possible.
4. MC is a direct simulation of what happens physically.
5. We can also generate events weighted by f(x)

Hit-or-miss MC

- Define the function $g(x,y) = \begin{cases} 0 & \text{if } f(x) < y \\ 1 & \text{if } f(x) \geq y \end{cases}$
- then $\langle f \rangle = \dfrac{1}{n} \sum_{j=1}^{n} g(x_{2j-1}, x_{2j})$



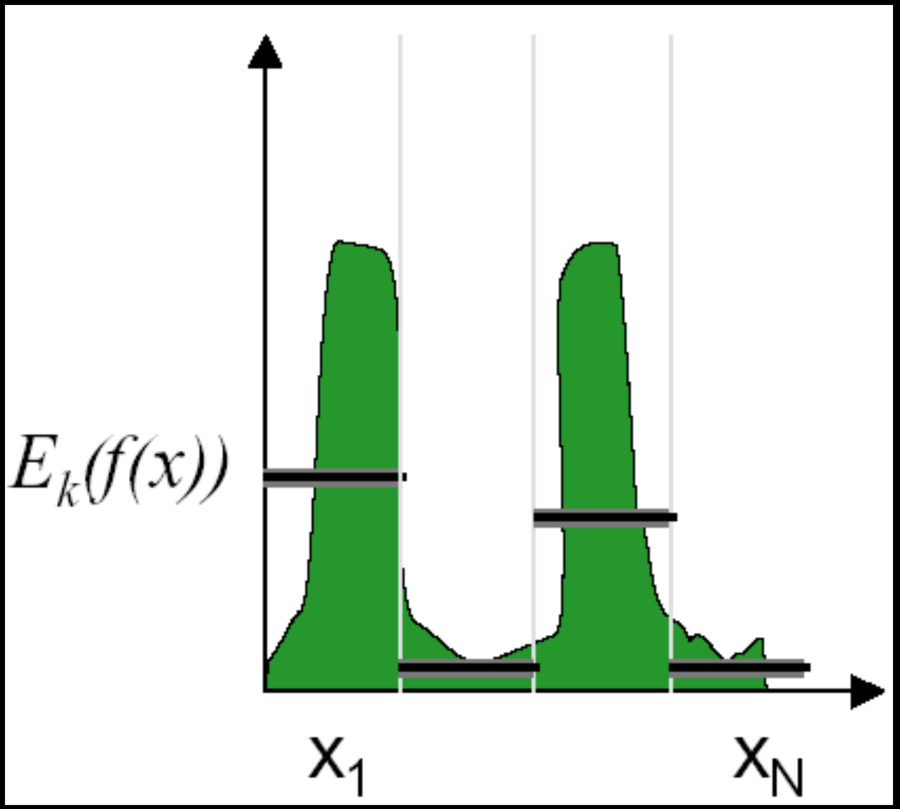
$$a = ymin = 0 \text{ and } b = ymax = 1$$

Stratified sampling

- just break the range of integration

$$0 = \alpha_0 < \alpha_1 \cdots < \alpha_k = 1$$

- apply crude MC to each interval

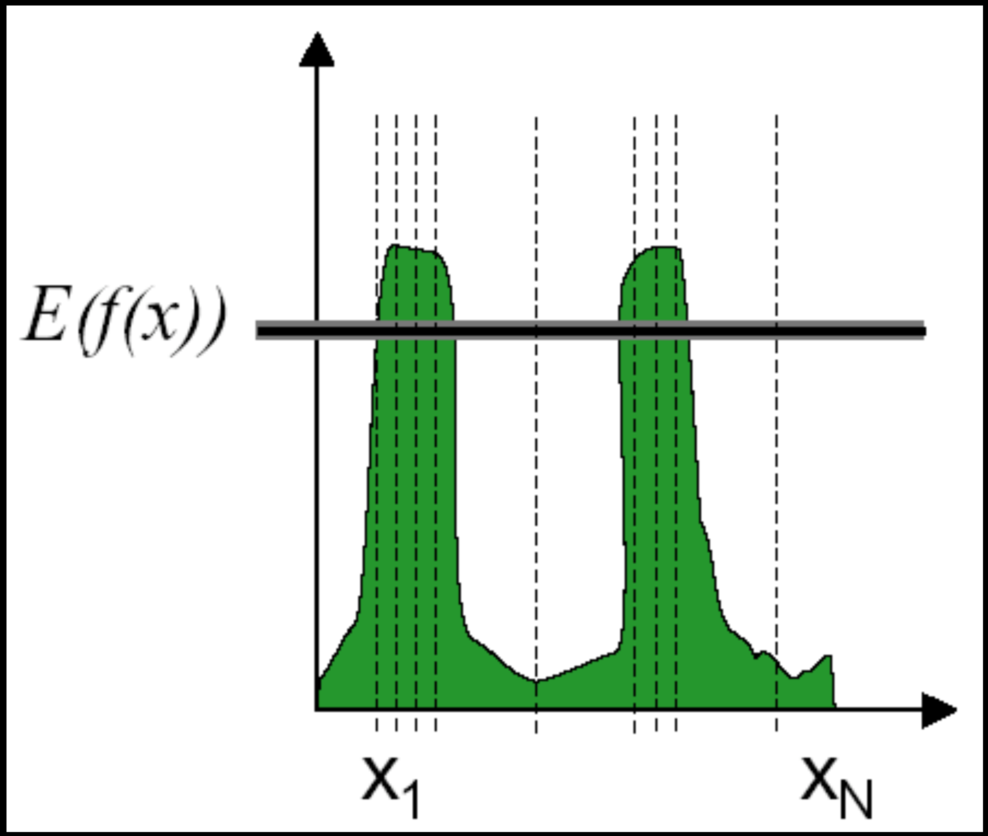


$$\langle f \rangle \simeq \sum_{j=1}^{k} (\alpha_j - \alpha_{j-1}) \frac{1}{n_j} \sum_{i=1}^{n_j} f(\alpha_{j-1} + (\alpha_j - \alpha_{j-1})\, x_{ij})$$

- variance is reduce for same number of calls of f.

Importance sampling

- use more points where the function is larger



- implementation using pdf's:

$$\langle f \rangle = \int_0^1 dx\ f(x) = \int_0^1 dx\ g(x)\ \frac{f(x)}{g(x)} = \int_0^1 dG\ \frac{f(x)}{g(x)}$$

where $$G(x) = \int_0^x dy\ g(y)$$

- choosing g(x) we can reduce the variance.

- the variance is $\sigma^2_{f/g} = \int_0^1 dG \left(\frac{f(x)}{g(x)} - \langle f \rangle \right)^2$

- g should be simple to obtain G explicitly
- if g=cf the variance vanishes
- choose a good function g similar to f

Particle Physics Applications

- Let's return to the cross section evaluation

$$\sigma = \int dx_1 dx_2 \sum_{\text{subp}} f_{a_1/p}(x_1)\, f_{a_2/\bar{p}}(x_2)$$

$$\frac{1}{2\hat{s}(2\pi)^{3n-4}} \int d\Phi_n(x_1 P_A + x_2 P_B;\, p_1 \ldots p_n)\Theta(\text{cuts})\overline{\sum}|\mathcal{M}|^2(a_1 a_2 \to b_1 \ldots b_n$$

that requires a suitable choice of the integration variables

- Initially we map the integration region into a 3n-2 hypercube

$$dx_1 dx_2 d\Phi_n = J \prod_{i=1}^{3n-2} dr_i$$

- It is easy to reconstruct the momenta and implement the cuts
- This procedure generate weighted events with weight

$$\mathbf{w} = \sum_{\{\mathbf{r_i}\}} \frac{\mathbf{J}}{\mathbf{2\hat{s}(2\pi)^{3n-4}}} \sum_{\mathbf{subprocesses}} \mathbf{f(x_1)f(x_2)}\overline{\sum}|\mathcal{M}|^2\, \mathbf{\Theta}(\mathrm{cuts}) \ ,$$

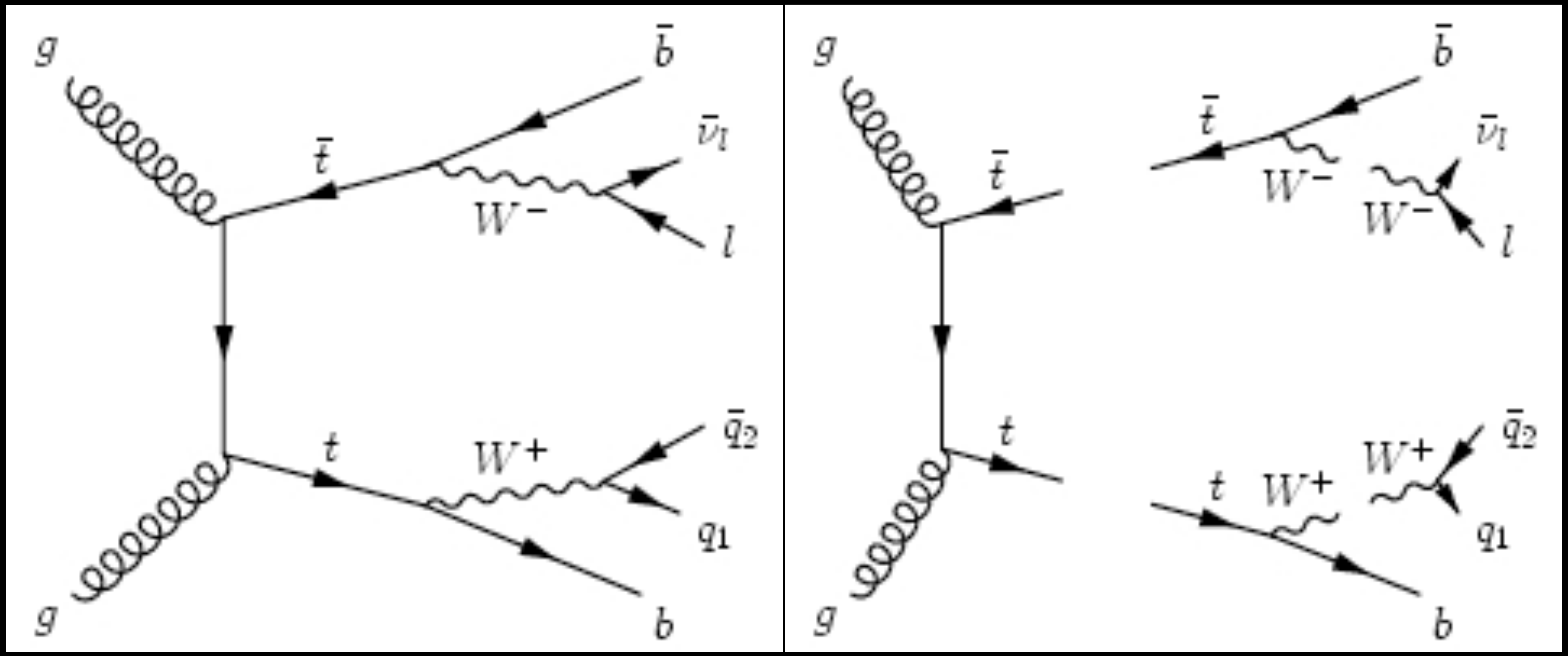
- Now it is possible to generate distributions
- Unweighted events can also be obtained

Phase space

- The sum over the final states leads to

$$d\Phi_n(ab \to 1 \ldots n) \equiv \delta^4(p_a + p_b - p_1 - \cdots - p_n) \prod_{i=1}^{n} \frac{d^3\vec{p}_i}{2E_i}$$

- this contains 3n-4 integrals
- Variables must be chosen to allow the improve the efficiency of MC
- we must have a feeling of the important contributions to the process



Two-body final state

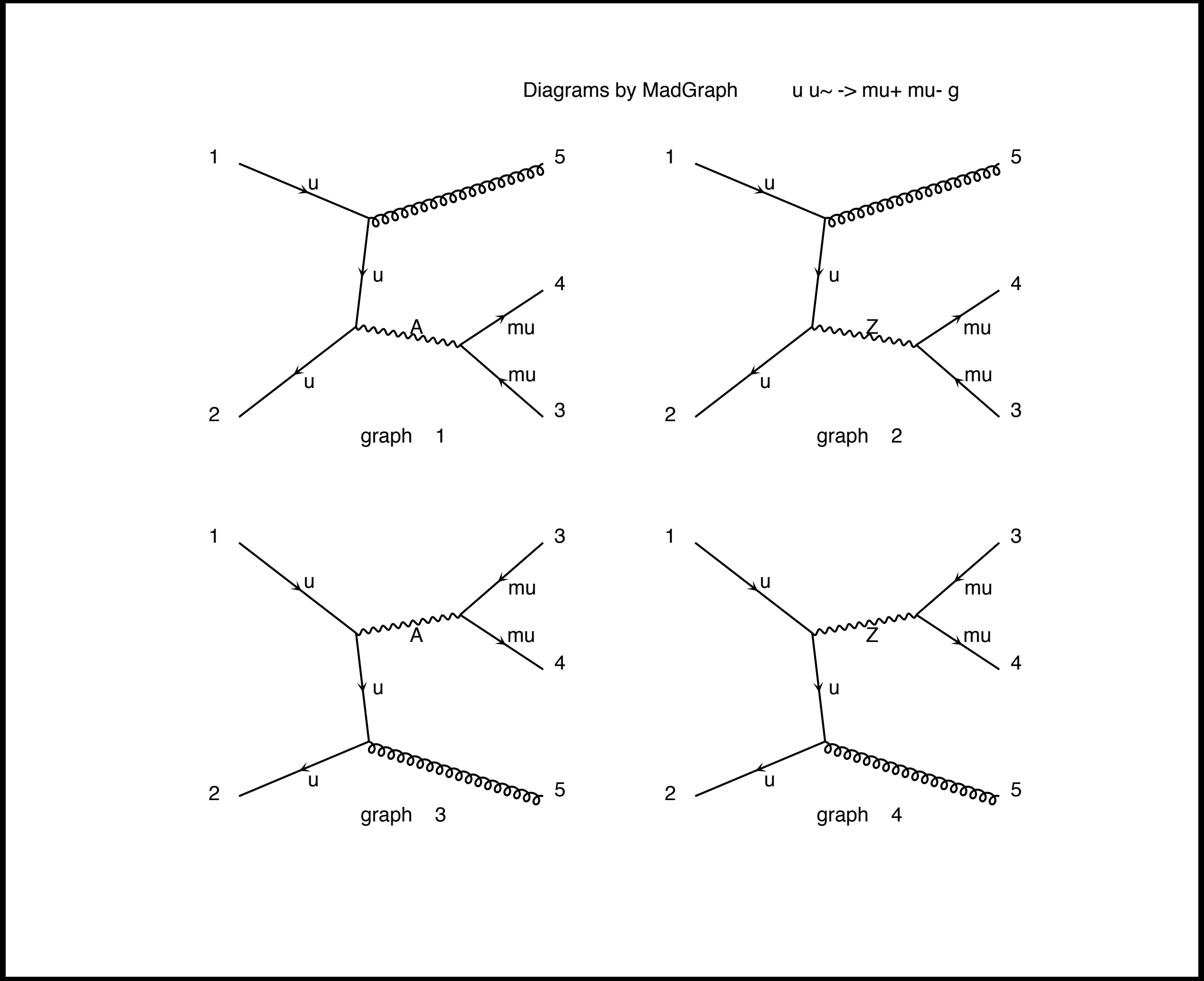
- this is the simplest possibility

$$
\begin{aligned}
d\Phi_2 &\equiv \delta^4\left(P - p_1 - p_2\right)\, \frac{d^3\vec{p}_1}{2E_1}\frac{d^3\vec{p}_2}{2E_2} \\
&= \frac{1}{4}\,\frac{|\vec{p}_1^{\,cm}|}{\sqrt{s}}\; d\Omega_1 = \frac{1}{4}\,\frac{|\vec{p}_1^{\,cm}|}{\sqrt{s}}\; d\cos\theta_1 d\phi_1 \\
&= \frac{1}{4}\,\frac{dt\ d\phi_1}{s\ \lambda^{1/2}\left(1, m_a^2/s, m_b^2/s\right)}
\end{aligned}
$$

with $\lambda(x,y,z) = (x-y-z)^2 - 4yz = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$

Multiparticle Phase Space

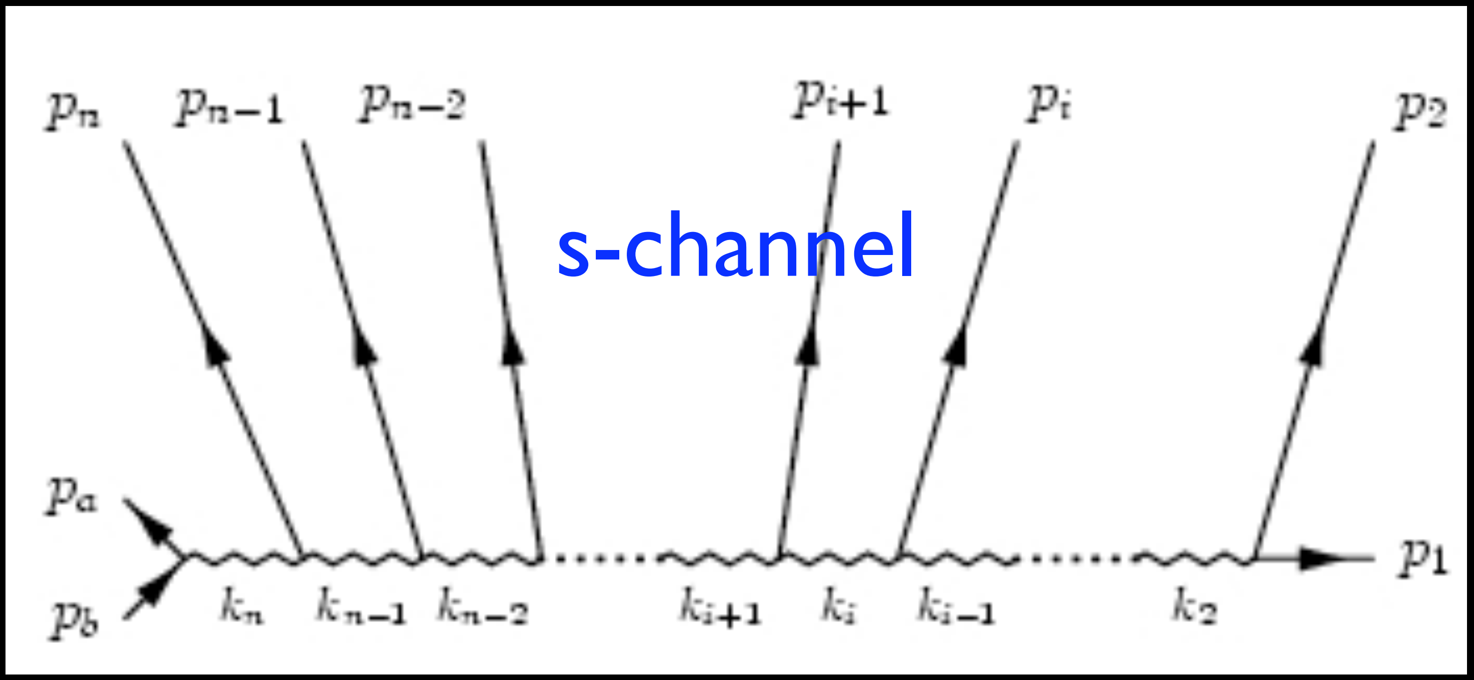
- Choice of variables decided by physics
- e.g., m34 in this example

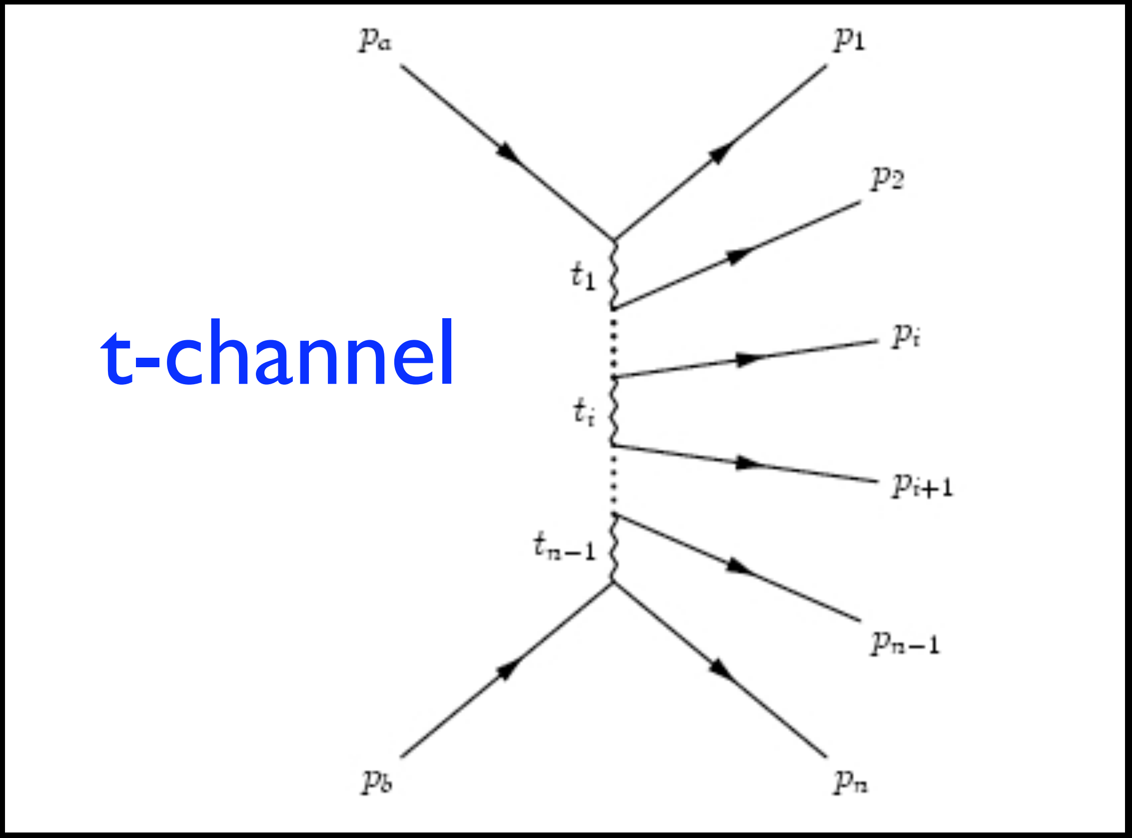


- a possible starting point is the relation: X=1+...j and Y=(j+1)+...+n

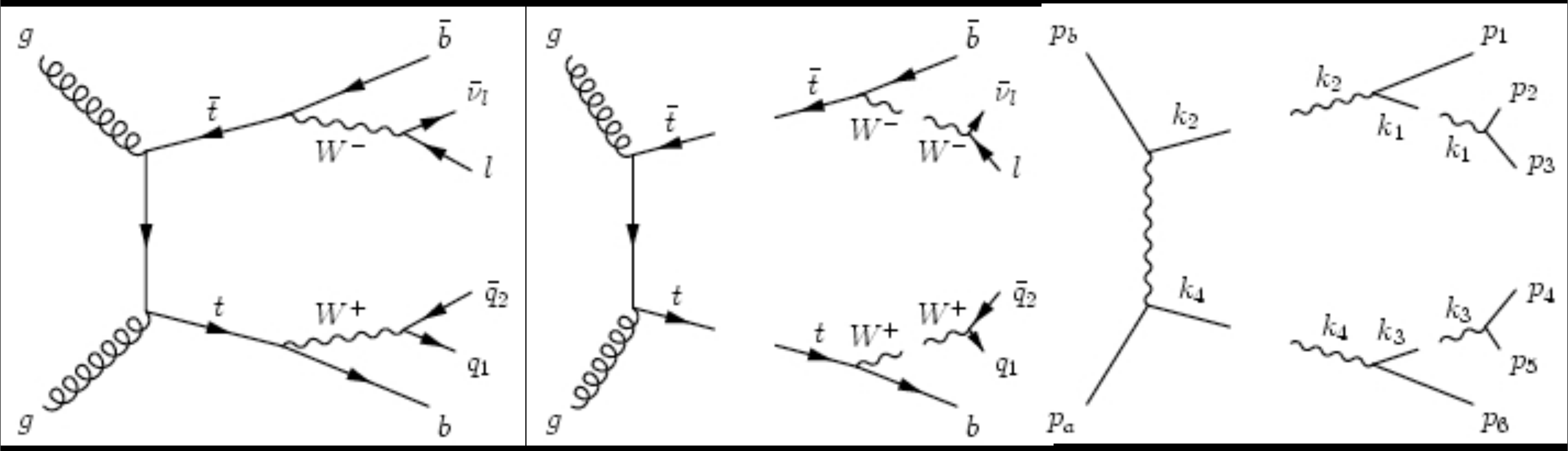
$$d\Phi_n(ab \to 1 \ldots n) = \quad d\Phi_2(ab \to XY) \times dM_X^2 \, dM_Y^2 \times d\Phi_j(X \to 1 \ldots j) \times d\Phi_{n-j}(Y \to j+1 \ldots n)$$

- it is possible to generate “generic” phase spaces





or more customized choices



Evaluation of scattering amplitudes

- We need to evaluate $|\mathcal{M}|^2(a_1 a_2 \to b_1 \ldots b_n)$ with $\mathcal{M} = \sum_{i=1}^{f} \mathcal{M}_i$

- In the trace technique we evaluate f(f+1)/2 terms like $\mathrm{Re}(\mathcal{M}_i^* \mathcal{M}_j)$

- Hard task for large f.
- Alternative evaluate numerically helicity amplitudes

$$|\mathcal{M}|^2 = \sum_{\lambda_a \ldots \lambda_n} |\mathcal{M}(\lambda_a \ldots \lambda_n)|^2$$

- MC gives a set of momenta from which we evaluate the matrix elements since they are just operations with matrices!

- Choose, for example, the representation

$$\text{for } \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{we write} \quad \psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} .$$

- the u spinors are given by $u(p,\sigma)_\pm = \sqrt{(p^0 \pm \sigma|\mathbf{p}|)}\ \chi_\sigma(p)$

$$\chi_+(p) = \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p_z)}} \begin{pmatrix} |\mathbf{p}|+p_z \\ p_x + ip_y \end{pmatrix} \ ; \ \chi_-(p) = \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}|+p_z)}} \begin{pmatrix} -p_x + ip_y \\ |\mathbf{p}|+p_z \end{pmatrix} .$$

- we can go on and define all elements in a Feynman diagram
- there are packages that do that for you, e.g., HELAS

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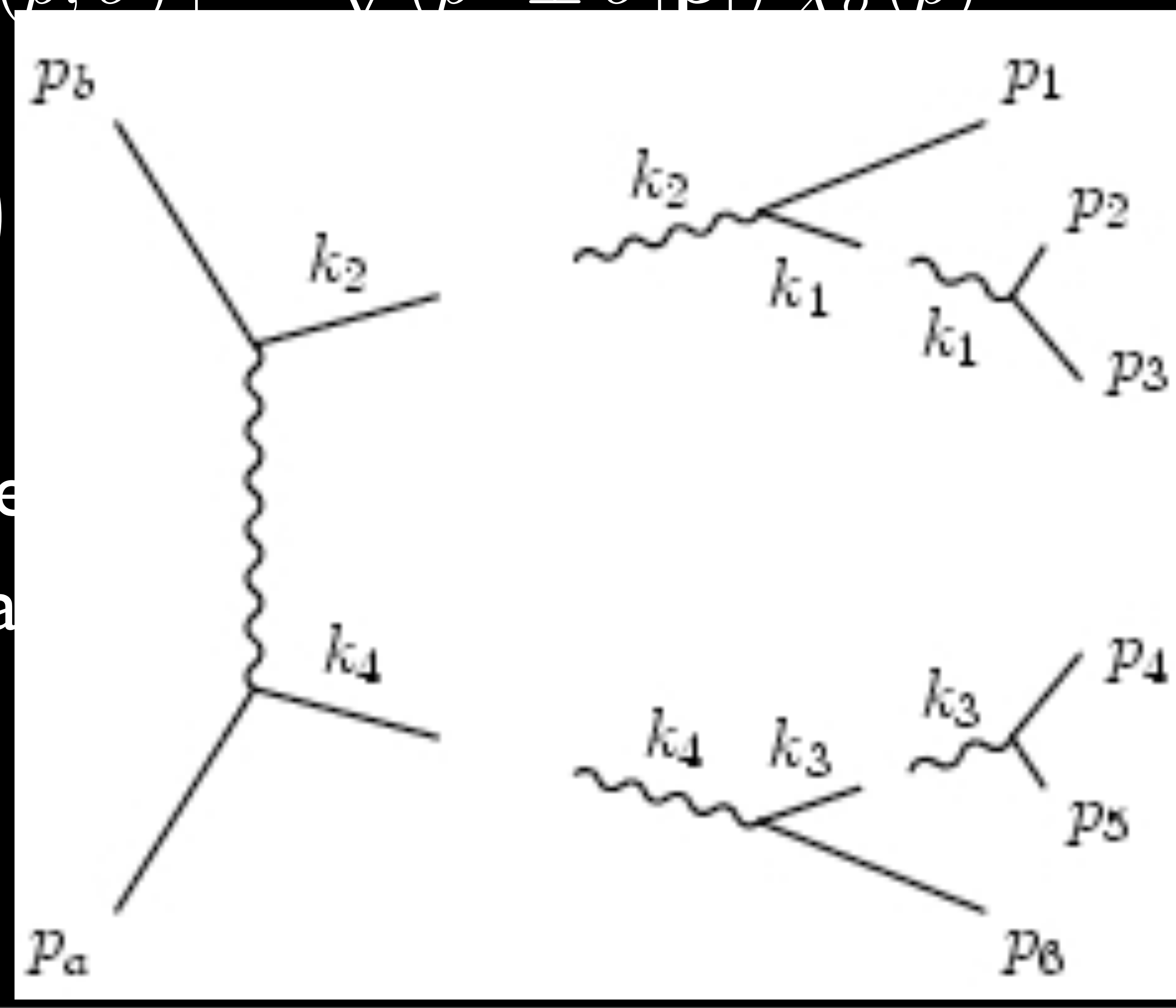


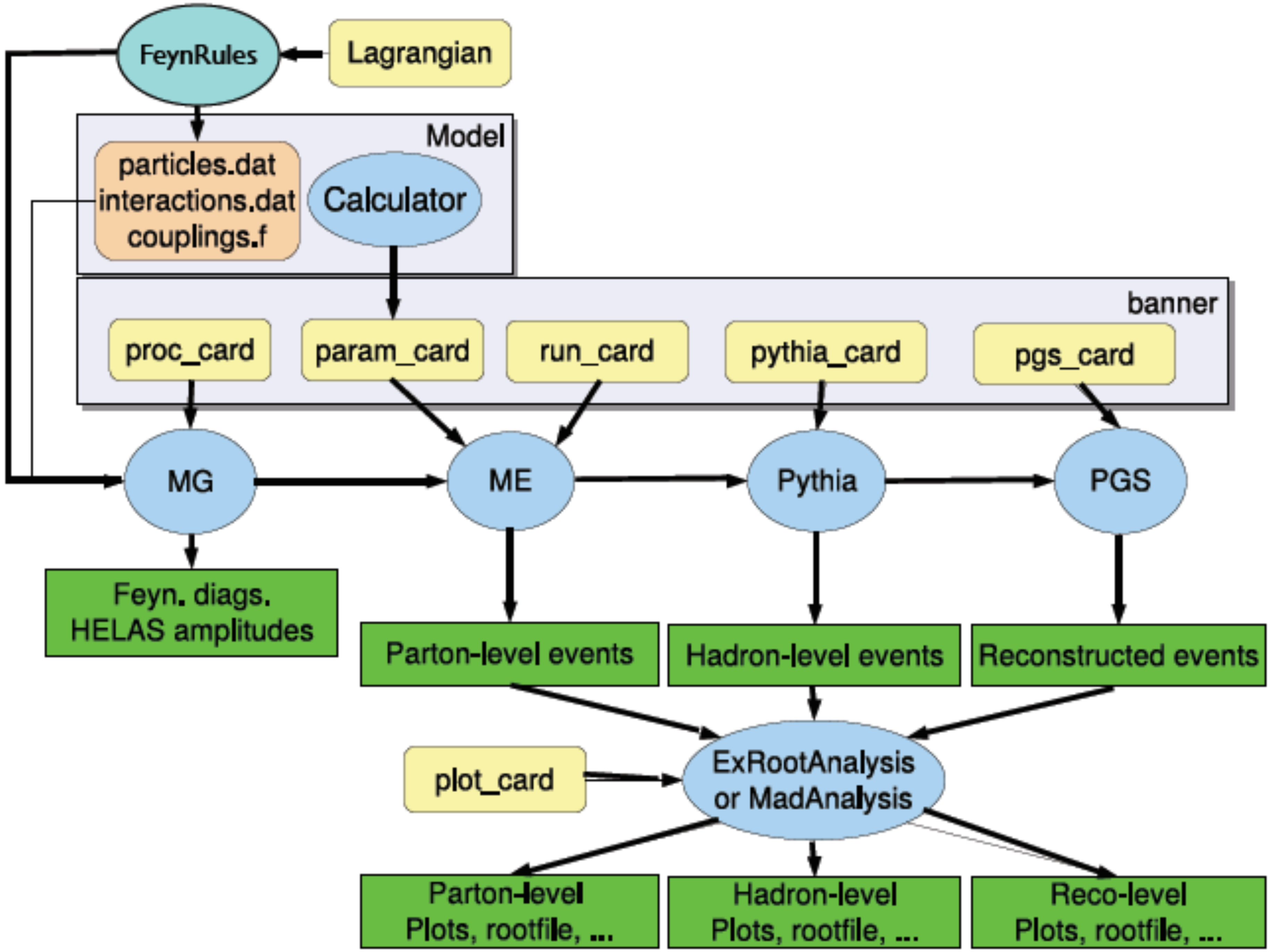
Madevent/Madgraph

- There are packages to evaluate the hard scattering part: MadGraph/MadEvent (ME), ALPGEN, CompHEP/CalcHEP, SHERPA, etc

- ME provides (automatically) complete partonic events:

1. Feynman diagrams;
2. Matrix element amplitudes;
3. Phase space integration;
4. Complete event simulation: MadEvent -> PYTHIA -> PGS
5. Merging with parton showers in PYTHIA
6. Available in the web

MadGraph/MadEvent v4 Flow

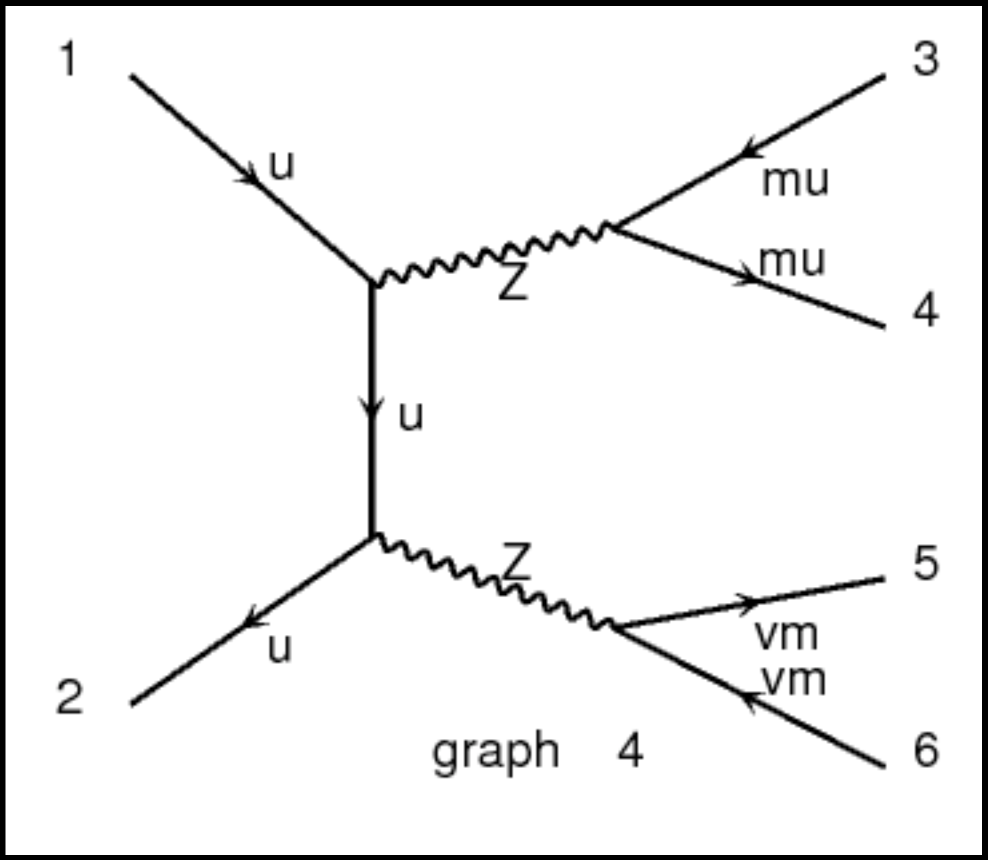


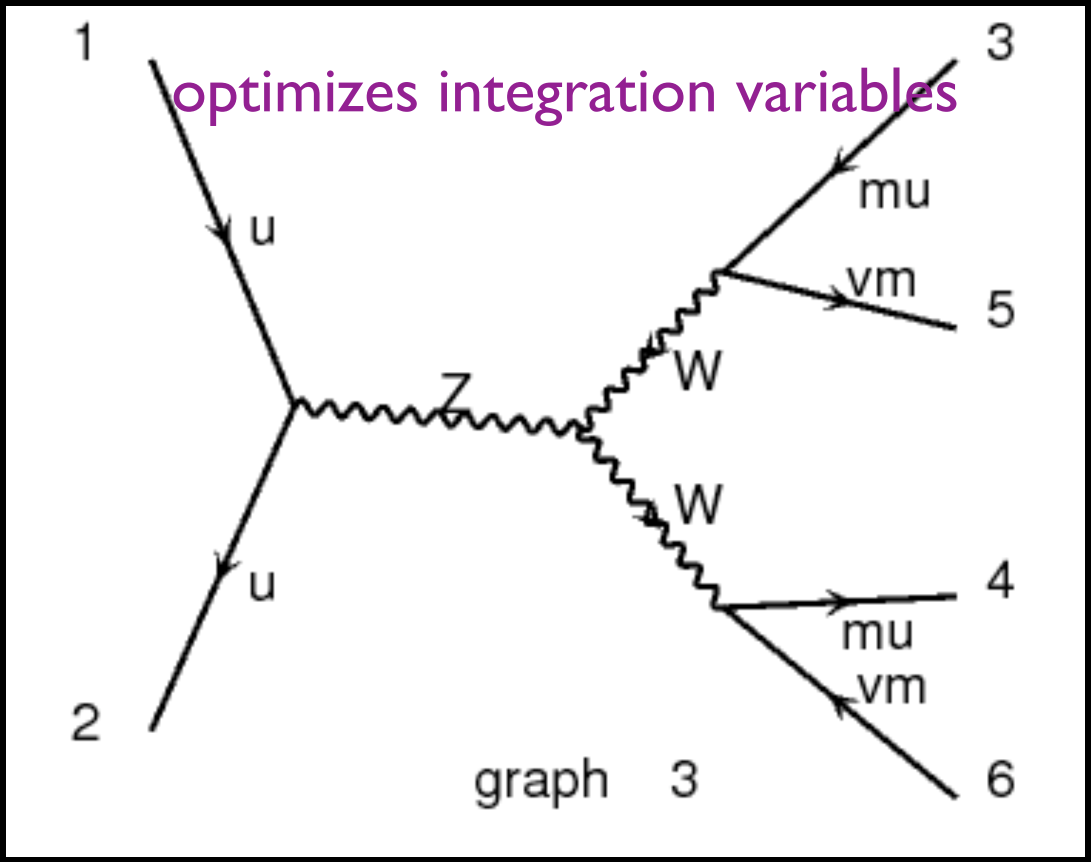
Phase space in MadEvent

- ME has a smart trick for the phase space

$$|M|^2 = \sum_{j=1}^{N} \frac{|M_j|^2}{\sum_k |M_k|^2} |M_1 + \cdots + M_N|^2$$

- Each element of the sum is dominated by a set of momenum configuration, so it chooses a different phase space for each j.

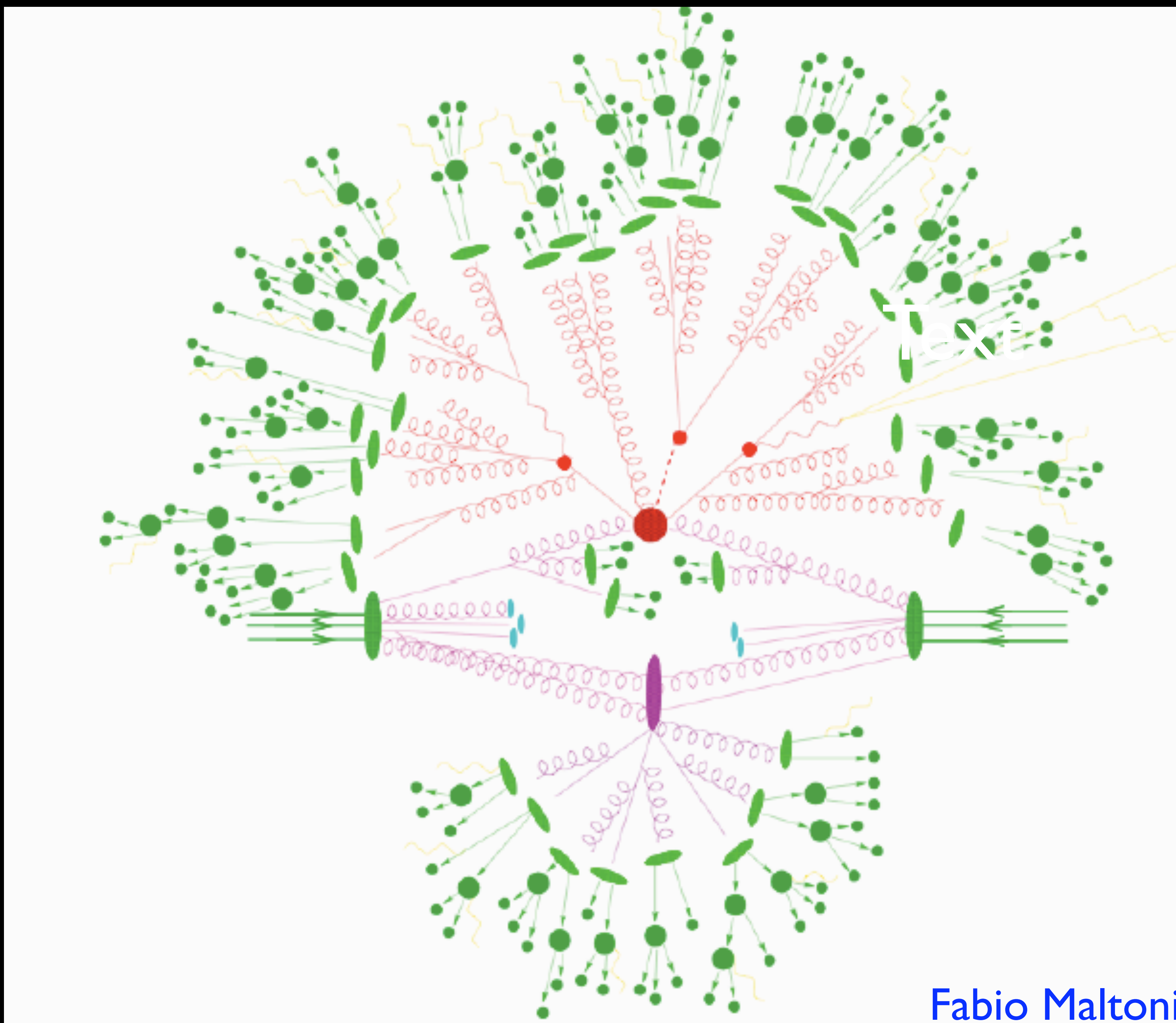




slows down the calculation for large number of Feynman diagrams

Beyond parton level processes

- Events in the LHA format passed for parton shower and hadronization by PYTHIA



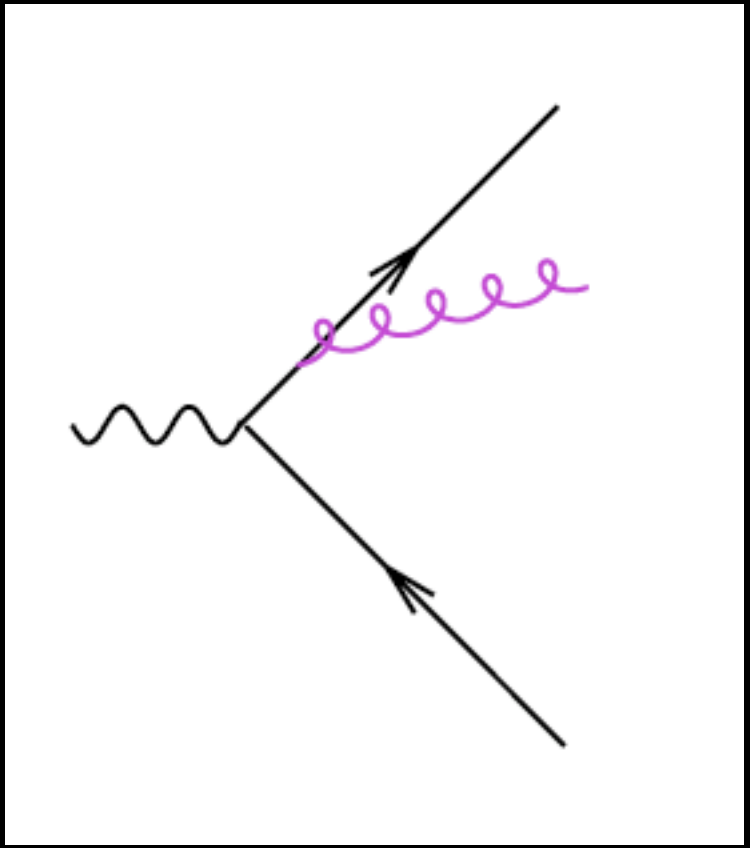
PS = approximation

hadronization = model

Parton shower basics

- matrix elements in g => q q g enhanced for g colinear to q

$$\frac{1}{(p_q+p_g)^2} \simeq \frac{1}{2E_gE_q(1-\cos\theta_{gq})}$$



there are soft and collinear divergences

dominant contribution to process

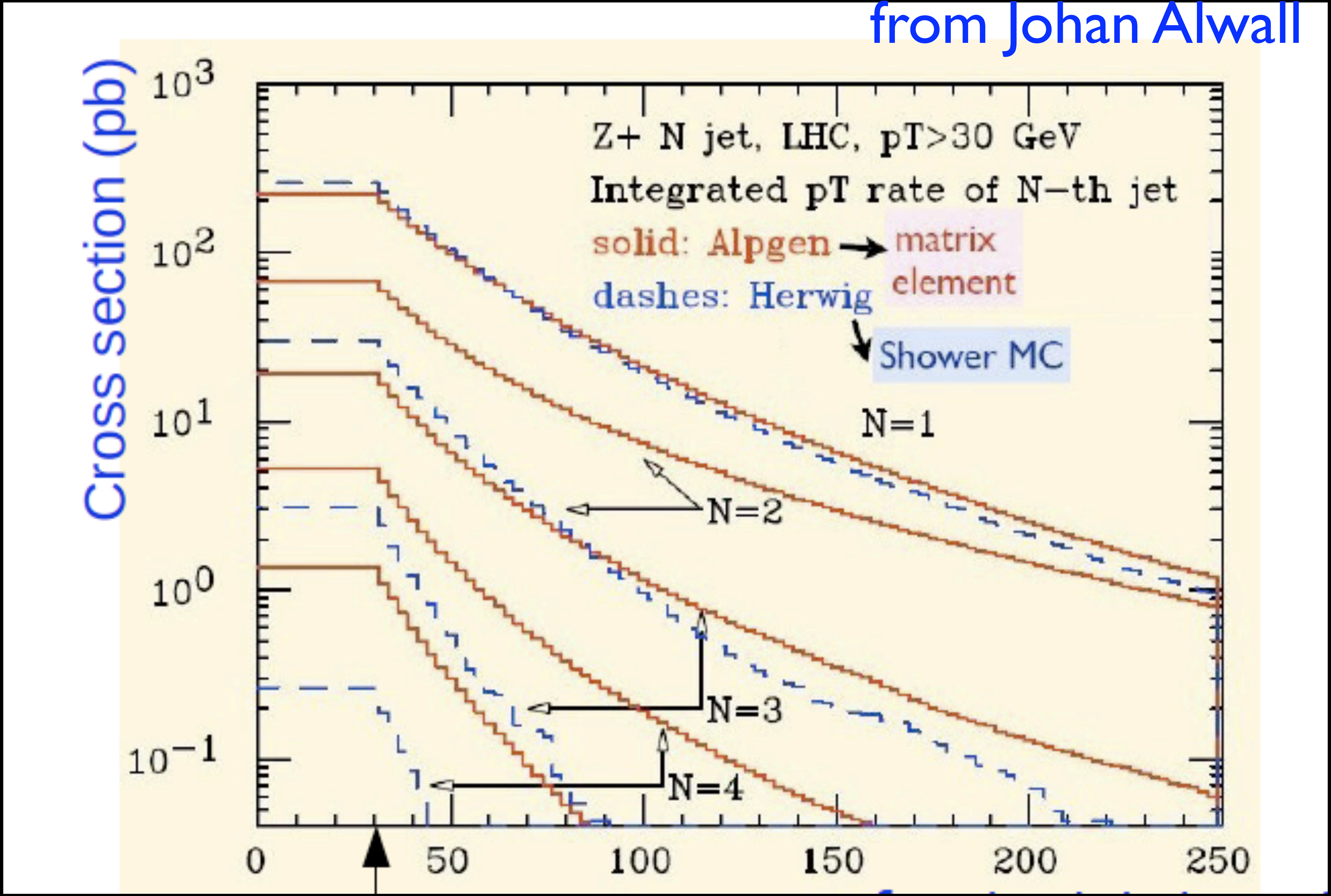
- Collinear factorization:

$$|M_{N+1}|^2 d\Phi_{N+1} \simeq |M_N|^2 d\Phi_N \frac{dt}{t} \frac{C\alpha_s}{2\pi} P(z)dz$$

- Parton shower resums leanding log contributions

Comments

- MC permeates all parts of simulation in HEP: hard scattering, parton shower, initial state radiation, hadronization...
- MC makes simples simulate cuts, build distributions
- Efficiency of the MC depends on the choice of variables
- MC also present in detector simulation
- Present hard scattering generators allows us to go beyond PS approximations. This is important!


from Johan Alwall
Cross section (pb)
Z+ N jet, LHC, pT>30 GeV
Integrated pT rate of N−th jet
solid: Alpgen
matrix element
dashes: Herwig
Shower MC
N=1
N=2
N=3
N=4
10^3
10^2
10^1
10^0
10^−1
0
50
100
150
200
250